

E L E M E N T S

O F

C O N I C S E C T I O N S,

&c.

ELEMENTS

CONSTITUTIONS



E L E M E N T S
OF *60 B 8*
C O N I C S E C T I O N S ;

W I T H
S E L E C T E X E R C I S E S

I N V A R I O U S B R A N C H E S O F
M A T H E M A T I C S A N D P H I L O S O P H Y .

F O R T H E U S E O F T H E R O Y A L M I L I T A R Y
A C A D E M Y A T W O O L W I C H .

BY CHARLES HUTTON, L.L.D. F.R.S.
PROFESSOR OF MATHEMATICS IN THE ROYAL MILITARY
ACADEMY. *K*

L O N D O N :

P R I N T E D B Y J . D A V I S .

Sold by G. G. J. and J. ROBINSON, Paternoster-Row.

MDCCLXXXVII.

✓



TO HIS GRACE

C H A R L E S,

DUKE OF RICHMOND, LENNOX, AND AUBIGNY,
&c. &c. &c.

&c. &c. &c.

MASTER GENERAL OF THE ORDNANCE.

MY LORD,

SHOULD this small performance continue to prove useful to the Institution for which it was composed, or become acceptable to the public; its existence, as well as its publication, must be ascribed to YOUR GRACE.

But, my LORD, the particular injunctions to avoid panegyric, under which permission for this dedication was granted, preclude me from explaining how the advanced state of learning in the Royal Military Academy ren-

A dered

A

dered

dered such a work necessary: since this would be to enumerate the means by which, under Your Auspices, this institution has attained a degree of perfection, which perhaps few public ones have ever equalled, none certainly have exceeded.

To YOUR GRACE therefore, with all Respect, this work is most humbly dedicated by,

MY LORD,

YOUR GRACE's

most obedient,

most devoted, and

most humble servant,

CHARLES HUTTON.

Royal Mil. Acad.
Aug. 24, 1787.

P R E F A C E.

THE want of a proper set of exercises, applied to the different branches of mathematical knowledge, which are deemed requisite to the military profession, induced me to draw up the following sheets. Their utility to the students in the Royal Military Academy, having been fully established, recommended them to the consideration of the Master General; and His Grace has been pleased to order them to be printed.

The miscellaneous form of this small work, arises from its consisting chiefly of practical questions in most of the sciences now taught in the Academy. Although Invention was not my immediate object, yet throughout the whole there will be found many things that are new, in

point of matter, but more so in the manner of treating the subjects.

In the Conic Sections there are several new and important properties dispersed through the books; and it is presumed that this branch is treated in a way better adapted to its intended use, than heretofore. The propositions, although demonstrated in a manner strictly geometrical, have this peculiarity, that only the first property of each section is demonstrated from the cone itself, and all the subsequent ones derived from the first, or from each other, in an easy and natural way, without introducing any arbitrary organical description of curves in plano. The arrangement of the steps in separate lines is also found to be an advantage, as it renders the demonstrations more easy to be comprehended, by presenting the whole to the eye in one connected view. And another very considerable improvement arises from the application of a new and general property concerning the intersections of a right line, with any of the curves, in two points: by means of which the general properties of the oblique ordinates, to any diameter, are easily deduced, without the forced and perplexed consideration of the areas of certain spaces.

The several propositions and properties, in the three curves or sections, are also arranged in such
4 order,

order, and enunciated in such manner, as to shew which properties are common to the different sections; and in particular it will be found, that most of the definitions and scholia are common to all the three curves; and that all the propositions and demonstrations of the ellipse, are literally the same with those of the hyperbola: a circumstance which must render both the learning and the remembering of the properties much easier than heretofore.

The collection of practical questions, which follow the Conic Sections, are mostly given without solutions, their answers only being set down, as probationary exercises to certain rules and branches of science contained in most books relating to these subjects. But the last collection, concerning forces, and the accompanying circumstances of time, space, and the velocity generated, have solutions annexed to them; as they require a knowledge of some other principles besides those that are usually found in the common books of science. Many of the problems in this part may, indeed, be met with elsewhere; but it is presumed, that the solutions will be found in general, either new, or attended with considerable improvement.

I have taken the liberty also to enrich this part with some new and useful problems relating
to

to the times of filling and emptying the ditches of fortifications, or other receptacles, with water, entering and evacuating them under certain circumstances. These curious problems, His Grace the Master General of the Ordnance was pleased to propose at a late examination of the Gentlemen Cadets; and the solutions at large, of such important propositions, are here published, as far as I know, for the first time.

To these succeeds the common theory of the motion of bodies in resisting mediums, but delivered in a manner, I trust, better adapted to practical purposes than in any former publication.

The volume then concludes with a compendium of some experiments, lately made to ascertain the actual resistance of the air to given surfaces, moving through it with given velocities, and different degrees of inclination: experiments, which it is to be wished may be farther prosecuted, as it is by such means only that the true theory of military projectiles, as well as other branches of natural philosophy, can be improved to any degree of practical utility.

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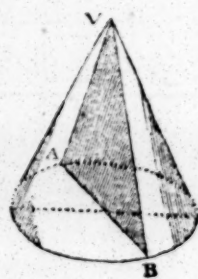
CONIC SECTIONS.

DEFINITIONS.

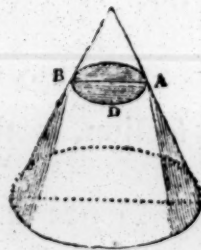
1. **C**ONIC sections are the figures made by the mutual intersection of a cone and a plane.

2. According to the different positions of the cutting plane, there arise five different figures or sections, namely, a triangle, a circle, an ellipse, a parabola, and an hyperbola : the three last of which only are peculiarly called conic sections.

3. If the cutting plane pass through the vertex of the cone, and any part of the base, the section will evidently be a triangle ; as VAB.



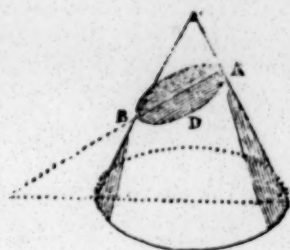
4. If the plane cut the cone parallel to the base, or make no angle with it, the section will be a circle ; as ABD.



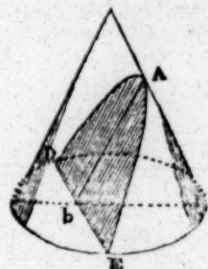
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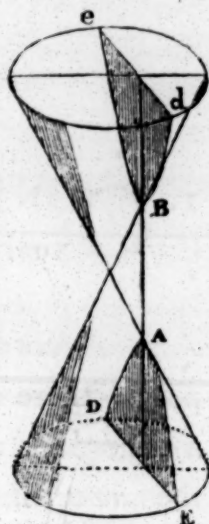
5. The section DAB is an ellipse, when the cone is cut obliquely through both sides, or when the plane is inclined to the base in a less angle than the side of the cone is.



6. The section is a parabola, when the cone is cut by a plane parallel to the base, or when the cutting plane and the side of the cone make equal angles with the base.



7. The section is an hyperbola, when the cutting plane makes a greater angle with the base than the side of the cone makes.



8. And if all the sides of the cone be continued through the vertex, forming an opposite equal cone, and the plane be also continued to cut the opposite cone, this latter section will be the opposite hyperbola to the former; as dbe .

9. The vertices of any section, are the points where the cutting plane meets the opposite sides of the cone, or the sides of the vertical triangular section; as A and B .

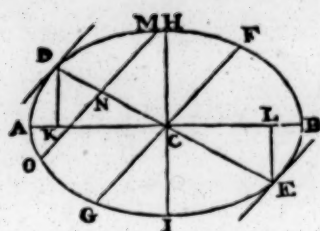
Hence the ellipse and the opposite hyperbolas, have each two vertices; but the parabola only one; unless we consider the other as at an infinite distance.

10. The Axis, or Transverse Diameter, of a conic section, is the line or distance AB between the vertices.

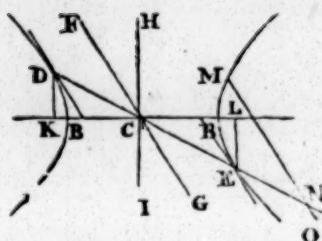
Hence the axis of a parabola is infinite in length, Ab being only a part of it.

11. The

Ellipse.



Oppos. Hyperb.



Parabola;



11. The Center c is the middle of the axis.

Hence the center of a parabola is infinitely distant from the vertex. And of an ellipse the axis and center lie within the curve; but of an hyperbola without.

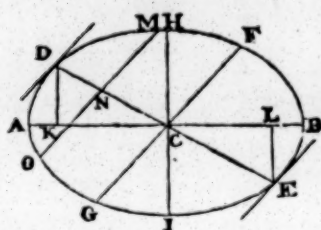
12. A Diameter is any right line, as AB or DE , drawn through the center, and terminated on each side by the curve; and the extremities of the diameter, or its intersections with the curve, are its vertices.

Hence all the diameters of a parabola are parallel to the axis, and infinite in length. And hence also every diameter of the ellipse and hyperbola have two vertices; but of the parabola only one; unless we consider the other as at an infinite distance.

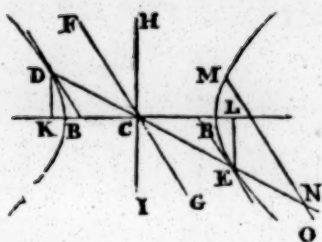
13. The Conjugate to any diameter, is the line drawn through the center, and parallel to the tangent of the curve at the vertex of the diameter. So FG , parallel to the tangent at D , is the conjugate to DE ; and HI , parallel to the tangent at A , is the conjugate to AB .

Hence the conjugate HI , of the axis AB , is perpendicular to it.

Ellipse.



Oppos. Hyperb.



Parabola.



14. An Ordinate to any diameter, is a line parallel to its conjugate, or to the tangent at its vertex, and terminated by the diameter and curve. So DK , EL are ordinates to the axis AB ; and MN , NO ordinates to the diameter DE .

Hence the ordinates to the axis are perpendicular to it.

15. An Absciss is a part of any diameter contained between its vertex and an ordinate to it; as AK or BK , or DN or EN .

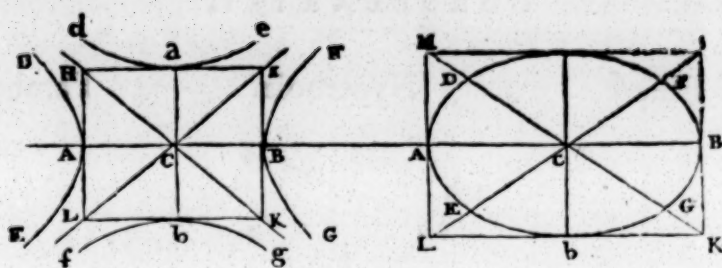
Hence, in the ellipse and hyperbola, every ordinate has two abscisses; but in the parabola, only one; the other vertex of the diameter being infinitely distant.

16. The Parameter of any diameter, is a third proportional to that diameter and its conjugate.

17. The Focus is the point in the axis where the ordinate is equal to half the parameter. As K and L , where DK or EL is equal to the semi-parameter.

Hence, the ellipse and hyperbola have each two foci; but the parabola only one.

18. If DAE , FBG be two opposite hyperbolas having AB for their first or transverse axis, and ab for their second or con-



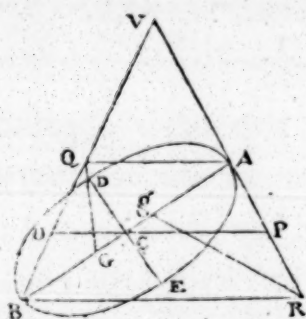
conjugate axis. And if dae , fbg be two other opposite hyperbolas having the same axes, but in the contrary order, namely, ab their first axis, and AB their second; then these two latter curves dae , fbg , are called the conjugate hyperbolas to the two former DAE , FBG ; and each pair of opposite curves mutually conjugate to the other.

19. And if tangents be drawn to the four vertices of the curves, or extremities of the axes, forming the inscribed rectangle $HIKL$; the diagonals HCK , ICL of this rectangle, are called the asymptotes of the curves.

SCHOLIUM.

The rectangle inscribed between the four conjugate hyperbolas, is similar to a rectangle circumscribed about an ellipse by drawing tangents, in like manner, to the four extremities of the two axes; and the asymptotes or diagonals in the hyperbola, are analogous to those in the ellipse, cutting this curve in similar points, and making the pair of equal conjugate diameters. Moreover, the whole figure, formed by the four hyperbolas, is, as it were, an ellipse turned inside out, cut open at the extremities D , E , F , G , of the said equal conjugate diameters, and those four points drawn out to an infinite distance, the curvature being turned the contrary way, but the axes, and the rectangle passing through their extremities, continuing fixed.

Ellipse.



In like manner, in the hyperbola, the length of the semi-conjugate axis, CD or CE , is a mean proportional between CO and CP , drawn parallel to the base, and meeting the sides of the cone in O and P . Or, if AO' be drawn parallel to the side VB , and meet PC produced in O' , making $CO' = CO$; and on this diameter $O'P$ a circle be drawn parallel to the base: then the semi-conjugate CD or CE will be an ordinate of this circle, being perpendicular to $O'P$ as well as to AB .

DE² or 2C · 2E, or 2CO · 2CP, is = QA · BR, or
QA : DE :: DE : BR.

In the parabola both the transverse and conjugate are infinite; for AB and BR are both infinite.

COROL. 2. In all the sections AG will be equal to the parameter of the axis, if QG be drawn making the angle AQG equal to the angle BAR.

For, by the definition, $AB : DE :: DE : p$ the param.

But by corol. 1, $BR : DE :: DE : AQ$;

Therefore $AB : BR :: AQ : p$.

But, by similar triangles, $AB : BR :: AQ : AG$;

And therefore $AG = p$ the parameter.

In like manner BG will be equal to the parameter p , if RG be drawn to make the angle BRG = the angle ABQ; since here also $AB : AQ :: BR : BG = p$.

COROL. 3. Hence the upper hyperbolic section, or section of the opposite cone, is equal and similar to the lower section. For the two sections have the same transverse or first axis AB, and the same conjugate or second axis DE, which is the mean proportional between AQ and RB; they have also equal parameters AG, BG. So that the two opposite sections make, as it were, but the two opposite ends of one entire section or hyperbola, the two being every where mutually equal and similar. Like the two halves of an ellipse, with their ends turned the contrary way.

COROL. 4. And hence, although both the transverse and conjugate axis in the parabola be infinite, yet the former is infinitely greater than the latter, or has an infinite ratio to it. For the transverse has the same ratio to the conjugate, as the conjugate has to the parameter, that is, as an infinite to a finite quantity, which is an infinite ratio.

OF THE ELLIPSE.

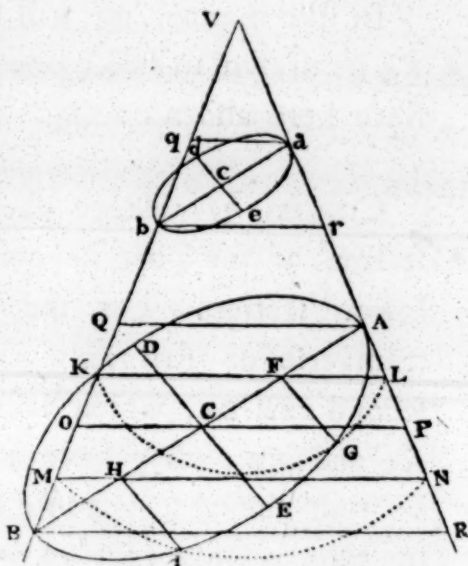
PROPOSITION I.

The Squares of the Ordinates of the Axis are to each other as the Rectangles of their Abscisses.

Let AVB be a plane passing through the axis of the cone; AGIH another section of the cone perpendicular to the plane of the former; AB the axis of this elliptic section; and FG, HI ordinates perpendicular to it. Then I say that

$$FG^2 : HI^2 :: AF \cdot FB : AH \cdot HB.$$

For, through the ordinates FG, HI draw the circular sections KGL, MIN parallel to the base of the cone, having KL, MN for their diameters, to which FG, HI are ordinates, as well as to the axis of the ellipse.



Now, by the similar triangles AFL, AHN, and BFL, BHM, we have $AF : AH :: FL : HN$,

and $FB : HB :: KF : MH$;

hence, taking the rectangles of the corresponding terms, we have the rect. $AF \cdot FB : AH \cdot HB :: KF \cdot FL : MH \cdot HN$.

But, by the nature of the circle, $KF \cdot FL = FG^2$, and $MH \cdot HN = HI^2$; Therefore the rect. $AF \cdot FB : AH \cdot HB :: FG^2 : HI^2$. Q.E.D.

COROL-

COROL. 1. All the parallel sections are similar figures, or have their two axes in the same proportion; that is, $AB : ab :: DE : de$.

For, by sim. triang. $AB : ab :: AQ : aq$,
and $AB : ab :: RB : rb$;

Theref. by comp. $AB^2 : ab^2 :: AQ \cdot RB : aq \cdot rb$.

But $AQ \cdot RB = DE^2$, and $aq \cdot rb = de^2$;

Therefore $AB^2 : ab^2 :: DE^2 : de^2$,
or $AB : ab :: DE : de$.

COROL. 2. Hence also, as the property is the same for the ordinates on both sides of the diameter, it follows, that

1st. At equal distances from the center, or from the vertices, the ordinates on both sides are equal, or that the double ordinates are bisected by the axis; and that the whole figure, made up of all the double ordinates, is also bisected by the axis.

2d. The two foci are equally distant from the center, or from either vertex.

COROL. 3. When the angle, which the plane of the section makes with the base of the cone, increases till it become equal to the angle made by the side of the cone and the base, or till the section be parallel to the opposite side of the base; then the axis becomes infinitely long, and the ellipse degenerates into a parabola; and because then the infinites FB and HB are in a ratio of equality, the general property,

namely $AF \cdot FB : AH \cdot HB :: FG^2 : HI^2$,

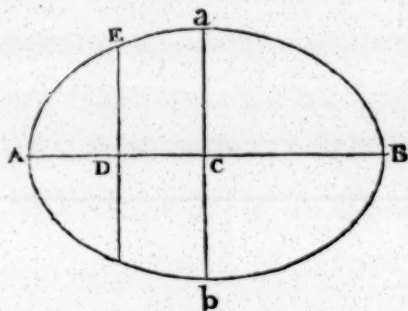
becomes $AF : AH :: FG^2 : HI^2$,

or, in the parabola, the abscissas are to each other, as the squares of their ordinates.

PROPOSITION II.

As the Square of the Transverse Axis :
 Is to the Square of the Conjugate ::
 So is the Rectangle of the Abscisses :
 To the Square of their Ordinate.

That is, $AB^2 : ab^2$ or $AC^2 : ac^2 :: AD \cdot DB : DE^2$.



For, by prop. I. $AC \cdot CB : AD \cdot DB :: ca^2 : DE^2$;
 But, if c be the center, then $AC \cdot CB = AC^2$, and ca is the semi-conj.
 Therefore $AC^2 : AD \cdot DB :: ac^2 : DE^2$;
 or, by permutation, $AC^2 : ac^2 :: AD \cdot DB : DE^2$;
 or, by doubling, $AB^2 : ab^2 :: AD \cdot DB : DE^2$. Q.E.D.

COROL. I. Or, because the rectangle $AD \cdot DB = CA^2 - CD^2$,
 the same property is $CA^2 : ca^2 :: CA^2 - CD^2 : DE^2$,
 or $AB^2 : ab^2 :: CA^2 - CD^2 : DE^2$.

COROL. 2. Or, by div. $AB : \frac{ab^2}{AB} :: CA^2 - CD^2 : DE^2$,
 that is, $AB : p :: AD \cdot DB$ or $CA^2 - CD^2 : DE^2$;
 where p is the parameter $\frac{ab^2}{AB}$ by the definition of it.

That

That is, As the transverse,
 Is to its parameter,
 So is the rectangle of the abscisses,
 To the square of their ordinate.

COROL. 3. When the axis AB is infinitely long, the curve becomes a parabola, and the infinities AB, DB are then in a ratio of equality; and then the last property,

namely $AB : p :: AD \cdot DB : DE^2$;

or $AB \cdot DE : AD \cdot DB :: p : DE$,

becomes $DE : AD :: p : DE$,

or $AD : DE :: DE : p$.

That is, in the parabola, the parameter is a third proportional to any abscissa and its ordinate.

PROPOSITION III.

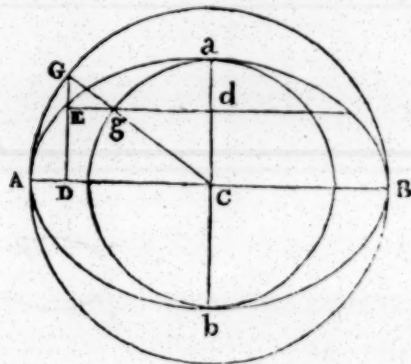
As the Square of the Conjugate Axis :

Is to the Square of the Transverse Axis ::

So is the Rectangle of the Abscisses of the Conjugate,
or the Difference of the Squares of the Semi-conjugate
and Distance of the Center from any Ordinate of that
Axis :

To the Square of that Ordinate.

That is, $ca^2 : cb^2 :: ad \cdot db$ or $ca^2 - cd^2 : de^2$.



For draw the ordinate ED to the transverse AB.

Then, by cor. I. prop. 2. $ca^2 : ca^2 :: ca^2 - cd^2 : de^2$.

But $cd^2 = de^2$, and $de^2 = cd^2$,

therefore $ca^2 : ca^2 :: ca^2 - de^2 : cd^2$,

or by alternation, $ca^2 : ca^2 - de^2 :: ca^2 : cd^2$,

and by division, $ca^2 : de^2 :: ca^2 : ca^2 - cd^2$,

and by alter. & inverf. $ca^2 : ca^2 :: ca^2 - cd^2 : de^2$. Q.E.D.

COROL.

COROL. 1. If two circles be described on the two axes as diameters, the one inscribed within the ellipse, and the other circumscribed about it; then an ordinate in the circle will be to the corresponding ordinate in the ellipse, as the axis of this ordinate, is to the other axis.

That is, $CA : ca :: DG : DE$,
and $ca : CA :: dg : dE$.

For, by the nature of the circle, $AD \cdot DB = DG^2$; theref. by the nature of the ellipse, $CA^2 : ca^2 :: AD \cdot DB^2$ or $DG^2 : DE^2$,
or $CA : ca :: DG : DE$.

In like manner $ca : CA :: dg : dE$.

Moreover, by equality, $DG : DE$ or $cd :: dE$ or $DC : dg$.

Therefore cgg is a continued strait line.

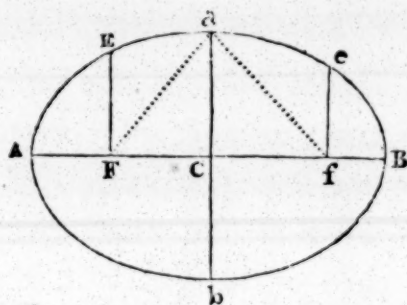
COROL. 2. Hence also, as the ellipse and circle are made up of the same number of corresponding ordinates, which are all in the same proportion of the two axes, it follows that the areas of the whole circle and ellipse, as also of any like parts of them, are in the same proportion of the two axes, or as the square of the diameter to the rectangle of the two axes; that is, the areas of the two circles, and of the ellipse, are as the square of each axis and the rectangle of the two, and therefore the ellipse is a mean proportional between the two circles.

PROPOSITION IV.

The Square of the Distance of the Focus from the Center, is equal to the Difference of the Squares of the Semi-axis; Or, the Square of the Distance between the Foci, is equal to the Difference of the Squares of the two Axes.

$$\text{That is, } CF^2 = CA^2 - ca^2,$$

$$\text{or } Ff^2 = AB^2 - ab^2.$$



For, to the focus F draw the ordinate FE ; which, by the definition, will be the semi-parameter. Then by the nature of the curve $CA^2 : ca^2 :: CA^2 - CF^2 : FE^2$; and by the def. of the para. $CA^2 : ca^2 :: ca^2 : FE^2$; therefore $ca^2 = CA^2 - CF^2$; and by addit. and subtr. $CF^2 = CA^2 - ca^2$; or, by doubling, $Ff^2 = AB^2 - ab^2$. Q.E.D.

COROL. I. The two semi-axes, and the focal distance from the center, are the sides of a right angled triangle cFa ; and the distance Fa from the focus to the extremity of the conjugate axis, is $= AC$ the semi-transverse.

For

For, as above, $CA^2 - ca^2 = CF^2$,
 and by right angled triangles $Fa^2 - ca^2 = CF^2$,
 therefore $CA = Fa$, and $AB = Fa + fa$.

COROL. 2. The conjugate semi-axis ca is a mean proportional between AF , FB , or between Af , fB , the distances of either focus from the two vertices.

For $ca^2 = CA^2 - CF^2 = CA + CF \cdot CA - CF = AF \cdot FB$.

COROL. 3. The same rectangle $AF \cdot FB$ of the focal distances from either vertex, is also equal to the rectangle $AC \cdot FE$ under the semi-transverse and its semi-parameter; since this last is equal to the square of the semi-conjugate by the definition of the parameter.

Or $AF : FE :: AC : FB$.

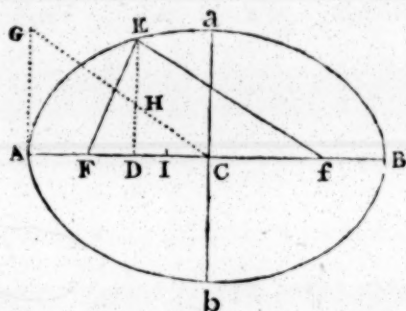
PROPOSITION V.

The Difference between the Semi-transverse and a Line drawn from the Focus to any Point in the Curve, is equal to a Fourth Proportional to the Semi-transverse, the Distance from the Center to the Focus, and the Distance from the Center to the Ordinate belonging to that Point of the Curve.

That is, $AC - FE = CI$, or $FE = AI$;

and $FE - AC = CI$, or $FE = BI$.

Where $CA : CF :: CD : CI$ the 4th proportional to CA , CF , CD .



For, by right angled triangles, $FE^2 = FD^2 + DE^2$.

Now draw AG parallel and equal to ca the semi-conjugate ; and join CG meeting the ordinate DE in H .

Then, by prop. 2, $CA^2 : AG^2 :: CA^2 - CD^2 : DE^2$;

and, by sim. tri. $CA^2 : AG^2 :: CA^2 - CD^2 : AG^2 - DH^2$;

consequently $DE^2 = AG^2 - DH^2 = ca^2 - DH^2$.

Also $FD = CF - CD$, and $FD^2 = CF^2 - 2CF \cdot CD + CD^2$;

therefore $FE^2 = CF^2 + ca^2 - 2CF \cdot CD + CD^2 - DH^2$.

But

But by prop. 4. $ca^2 + CF^2 = CA^2$
 and, by supposition, $2CF \cdot CD = 2CA \cdot CI$;
 theref. $FE^2 = CA^2 - 2CA \cdot CI + CD^2 - DH^2$.

But, by supposition $CA^2 : CD^2 :: CF^2$ or $CA^2 - AG^2 : CI^2$;
 and, by sim. tri. $CA^2 : CD^2 :: CA^2 - AG^2 : CD^2 - DH^2$;
 therefore $CI^2 = CD^2 - DH^2$;

consequently $FE^2 = CA^2 - 2CA \cdot CI + CI^2$.

And the root or side of this square is $FE = CA - CI = AI$.

In the same manner is found $fE = CA + CI = BI$. Q.E.D.

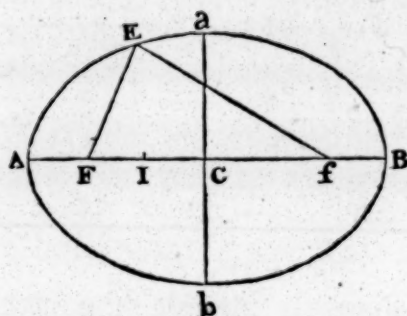
COROL. 1. Hence CI or $CA - FE$ is a 4th proportional to CA, CF, CD .

COROL. 2. And $fE - FE = 2CI$; that is, the difference between two lines drawn from the foci, to any point in the curve, is double the 4th proportional to CA, CF, CD .

PROPOSITION VI.

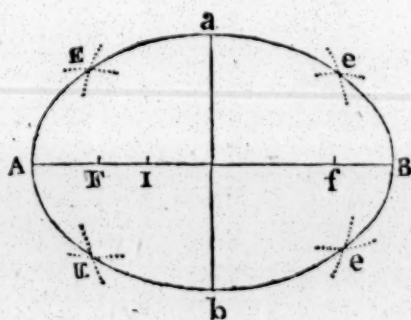
The Sum of two Lines drawn from the Foci, to meet in any Point of the Curve is equal to the Transverse Axis.

That is, $FE + fE = AB$.



For, by the last prop. $FE = CA - CI = AI$,
and, by the same, $fE = CA + CI = BI$;
theref. by addition, $FE + fE = AB$.

COROL. Hence is derived the common method of describing the curve mechanically by points, or with a thread, thus.



In

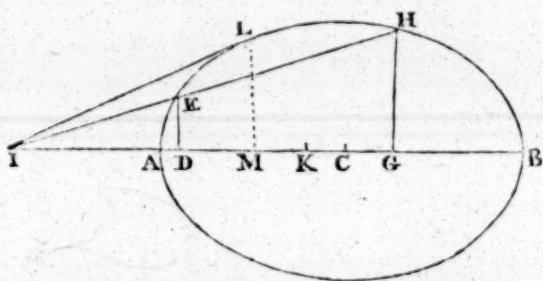
In the transverse take the foci F, f , and any point I . Then with the radii AI, BI , and centers F, f , describe arcs intersecting in E , which will be a point in the curve. In like manner, assuming other points I , as many other points will be found in the curve. Then with a steady hand, draw the curve line through all the points of intersection E .

Or, take a thread of the length of AB the transverse axis, and fix its two ends in the foci F, f , by two pins. Then carry a pen or pencil round by the thread, keeping it always stretched, and its point will trace out the curve line.

PROPOSITION VII.

If from any Point *I* in the Axis produced, a Line *IEH* be drawn cutting the curve in Two Points; and from those Two Points be drawn the Perpendicular Ordinates *DE*, *GH*; and if *K* be the Middle of *DG*, and *c* the Center or the Middle of *AB*: Then shall *CK* be to *CI* as the Rectangle of *AD* and *AG* to the Square of *AI*.

That is, $CK : CI :: AD \cdot AG : AI^2$.



For, by prop. I. $AD \cdot DB : AG \cdot GB :: DE^2 : GH^2$,
 and by sim. tri. $ID^2 : IG^2 :: DE^2 : GH^2$;
 theref. by equality, $AD \cdot DB : AG \cdot GB :: ID^2 : IG^2$.

But $DB = 2CK + AG$, and $GB = 2CK + AD$,

theref. $AD \cdot 2CK + AD \cdot AG : AG \cdot 2CK + AD \cdot AG :: ID^2 : IG^2$,
 and, by div. $DG \cdot 2CK : IG^2 - ID^2$ or $DG^2 \cdot 2IK : AD \cdot 2CK + AD \cdot AG :: ID^2$.

or $2CK : 2IK :: AD \cdot 2CK + AD \cdot AG : ID^2$

or $AD \cdot 2CK : AD \cdot 2IK :: AD \cdot 2CK + AD \cdot AG : ID^2$;

theref.

theref. by div. $CK : IK :: AD \cdot AG : ID^2 - AD \cdot 2IK,$

and, by comp. $CK : CI :: AD \cdot AG : ID^2 - AD \cdot ID + IA,$

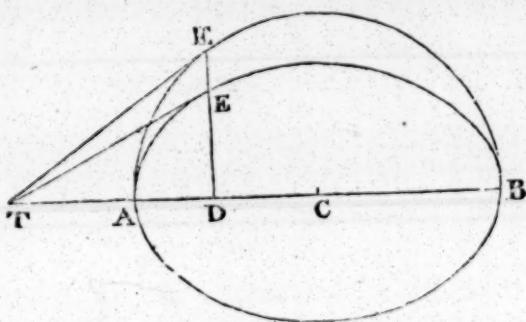
or $CK : CI :: AD \cdot AG : AI^2.$ Q.E.D.

COROL. When the line IH , by revolving about the point I , comes into the position of the tangent IL , and the ordinate LM being drawn, then the points E and H meet in the point L , and the points D, K, G , coincide with the point M ; and then the property in the proposition becomes $CM : CI :: AM^2 : AI^2.$

PROPOSITION VIII.

If a Tangent and Ordinate be drawn from any Point in the Curve, meeting the Transverse Axis; the Semi-transverse will be a Mean Proportional between the Distances of the said Two Intersections from the Center.

That is, CA is a mean proportion between CD and CT ; or CD, CA, CT are continued proportionals.



For, by cor. prop. 7, $CD : CT :: AD^2 : AT^2$,
 that is, $CD : CT :: \overline{CA - CD}^2 : \overline{CT - CA}^2$,
 or $:: CD^2 + CA^2 : CA^2 + CT^2$,
 and $:: CD^2 : CA^2$,
 and $:: CA^2 : CT^2$;
 therefore $CD : CA :: CA : CT$. Q.E.D.

COROL. I. Since CT is always a third proportional to CD, CA ; if the points D, A , remain constant, then will the point T be constant also; and therefore all the tangents will meet in this point T , which are drawn from E , of every

every ellipse described on the same axis AB , where they are cut by the common ordinate DEE drawn from the point D .

COROL. 2. Hence a tangent is easily drawn to the curve, from any point, either in the curve or without it.

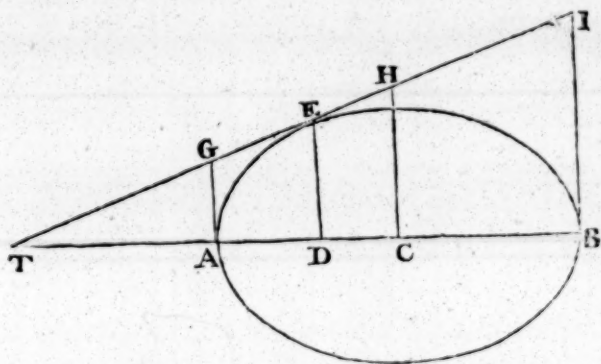
First, if the given point E be in the curve. Draw the ordinate DE of the diameter AC ; and in the diameter produced take CT a third proportional to CD , CA . Then join TE for the tangent required.

But if the point T be given any where without the curve. Join CT , in which take CD a third proportional to CT , CA ; and draw the ordinate DE . Then join TE as before.

PROPOSITION IX.

If there be any Tangent meeting Four Perpendiculars to the Axis drawn from these four Points, namely the Center, the two Extremities of the Axis, and the Point of Contact; those Four Perpendiculars will be Proportionals.

That is, $AG : DE :: CH : BI$.



For, by prop. 8, $TC : AC :: AC : DC$,
 theref. by div. $TA : AD :: TC : AC$ or CB ,
 and by comp. $TA : TD :: TC : TB$,
 and by sim. tri. $AG : DE :: CH : BI$.

Q.E.D.

COROL. Hence TA, TD, TC, TB } are also proportionals.
 and TG, TE, TH, TI }

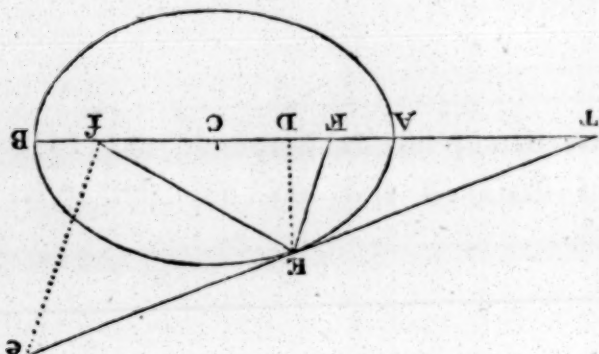
For these are as AG, DE, CH, BI , by similar triangles.

PRO-

PROPOSITION X.

If there be any Tangent, and two Lines drawn from the Foci to the Point of Contact; these two Lines will make equal Angles with the Tangent.

That is, the $\angle FET = \angle fee$.



For draw the ordinate DE, and fe parallel to FE.

By cor. I. prop. 5, $CA : CD :: CF : CA - FE$,
 and by prop. 8, $CA : CD :: CT : CA$;
 therefore $CT : CF :: CA : CA - FE$;
 and by add. and sub. $TF : Tf :: FE : 2CA - FE$ or FE by prop. 6.
 But by sim. tri. $TF : Tf :: FE : fe$;
 therefore $FE = fe$, and conseq. $\angle e = \angle fee$.
 But, because FE is parallel to fe , the $\angle e = \angle FET$;
 therefore the $\angle FET = \angle fee$. Q.E.D.

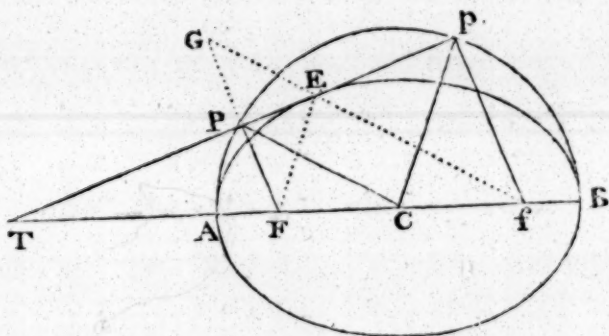
COROL. As opticians find that the angle of incidence is equal to the angle of reflection, it appears from our proposition, that rays of light issuing from the one focus, and meeting the curve in every point, will be reflected into lines drawn from the other focus. So the ray FE is reflected into FE . And this is the reason why the points F, f are called *foci*, or burning points.

PRO-

PROPOSITION XI.

If a Line be drawn from either Focus, Perpendicular to a Tangent to any Point of the Curve ; the Distance of their Interfection from the Center will be equal to the Semi-transverse Axis.

That is, if FP , fp be perpendicular to the Tangent TPP , then shall CP and cp be each equal to CA or CB .



For, through the point of contact E draw FE and fp meeting FP produced in G . Then, the $\angle GEP = \angle FEP$, being each equal to the $\angle fep$, and the angles at P being right, and the side PE being common, the two triangles GEP , FEP are equal in all respects, and so $GE = FE$, and $GP = FP$. Therefore, since $FP = \frac{1}{2} FG$, and $FC = \frac{1}{2} Ff$, and the angle at F common, the side CP will be $= \frac{1}{2} FG$ or $\frac{1}{2} AB$, that is $CP = CA$ or CB .

And in the same manner $cp = CA$ or CB .

Q.E.D.

COROL.

COROL. 1. A circle described on the transverse axis, as a diameter, will pass through the points P, p ; because all the lines CA, CP, cp, CB , being equal, will be radii of the circle.

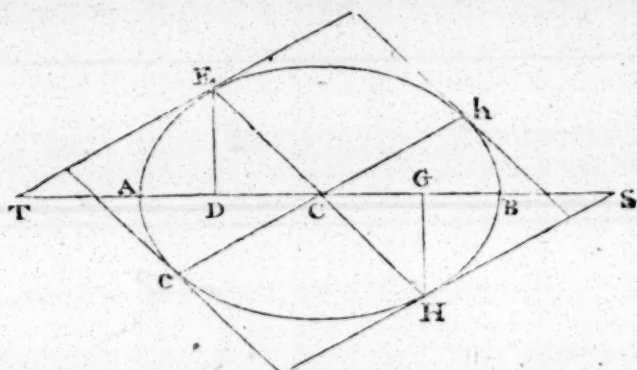
COROL. 2. CP is parallel to fE , and cp parallel to FE .

COROL. 3. If at the intersections of any tangent, with the circumscribed circle, perpendiculars to the tangent be drawn, they will meet the transverse axis in the two foci. That is, the perpendiculars PF, pf give the foci F, f .

PROPOSITION XII.

The equal Ordinates, or the Ordinates at equal Distances from the Center, on the opposite Sides and Ends of an Ellipse, have their Extremities connected by one Right Line passing through the Center, and that Line is bisected by the Center.

That is, if $CD = CG$, or the ordinate $DE = GH$; then shall $CE = CH$, and ECH will be a right line.



For, when $CD = CG$, then also is $DE = GH$ by cor. 2. prop. I. But the $\angle D = \angle G$, being both right angles; therefore the third side $CE = CH$, and the $\angle DCE = \angle GCH$, and consequently ECH is a right line.

COROL. I. And, conversely, if ECH be a right line passing through the center; then shall it be bisected by the center, or have $CE = CH$; also DE will be $= GH$, and $CD = CG$.

COROL.

COROL. 2. Hence also, if two tangents be drawn to the two ends E, H of any diameter EH they will be parallel to each other, and will cut the axis at equal angles, and at equal distances from the center. For, the two CD, CA being equal to the two CG, CB, the third proportionals CT, CS will be equal also; then the two sides CE, CT being equal to the two CH, CS, and the included angle ECT equal to the included angle HCS, all the other corresponding parts are equal: and so the $\angle T = \angle S$, and TE parallel to HS.

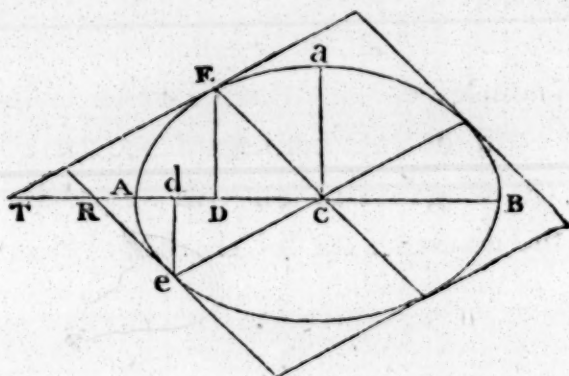
COROL. 3. And hence the four tangents, at the four extremities of any two conjugate diameters, form a parallelogram circumscribing the ellipse, and the pairs of opposite sides are each equal to the corresponding parallel conjugate diameters.

For, if the diameter eh be drawn parallel to the tangent TE or HS, it will be the conjugate to EH by the definition; and the tangents to eh will be parallel to each other, and to the diameter EH for the same reason.

PROPOSITION XIII.

If two Ordinates ED , ed be drawn from the Extremities E , e , of two Conjugate Diameters, and Tangents be drawn to the same Extremities, and meeting the Axis produced in T and R ;

Then shall CD be a mean Proportional between cd , dr ,
and cd a mean Proportional between CD , DT .



For, by prop. 8,	$CD : CA :: CA : CT$,	
and by the same	$cd : CA :: CA : CR$;	
theref. by equality	$CD : cd :: CR : CT$,	
But by sim. tri.	$DT : cd :: CT : CR$;	
theref. by equality	$CD : cd :: cd : DT$.	
In like manner	$cd : CD :: CD : dr$.	$Q.E.D.$

COROL. 1. Hence $CD : cd :: CR : CT$.

COROL. 2. Hence also $CD : cd :: de : DE$.

And the rect. $CD \cdot DE = cd \cdot de$, or $\Delta CDE = \Delta cde$.

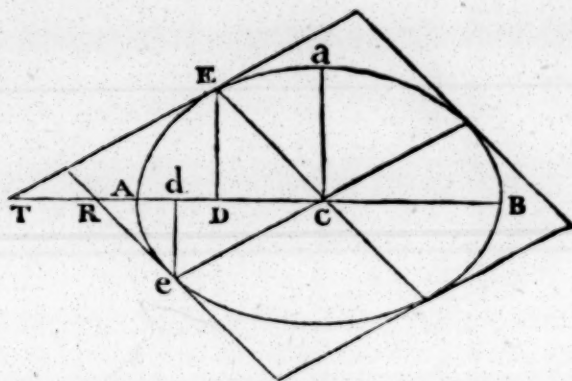
COROL. 3. Also $cd^2 = CD \cdot DT$,
and $CD^2 = cd \cdot dR$.

Or cd a mean proportional between CD , DT ;
and CD a mean proportional between cd , dR .

PROPOSITION XIV.

The same Figure being constructed as in the last Proposition, each Ordinate will divide the Axis, and the Semi-axis added to the external Part, in the same Ratio.

That is, $DA : DT :: DC : DB$,
and $dA : dR :: dC : dB$.



For, by prop. 8, $CD : CA :: CA : CT$,
 and by div. $CD : CA :: AD : AT$,
 and by comp. $CD : DB :: AD : DT$,
 or $DA : DT :: DC : DB$.
 In like manner $dA : dR :: dC : dB$.

Q.E.D.

COROL. 1. Hence, and from cor. 3 to the last prop. we have

$$\begin{aligned} \mathbf{CD}^2 &= \mathbf{CD} \cdot \mathbf{DT} = \mathbf{AD} \cdot \mathbf{DB} = \mathbf{CA}^2 - \mathbf{CD}^2, \\ \mathbf{CD}^2 &= \mathbf{cd} \cdot \mathbf{dR} = \mathbf{Ad} \cdot \mathbf{dB} = \mathbf{CA}^2 - \mathbf{cd}^2. \end{aligned}$$

COROL.

COROL. 2. Hence also $CA^2 = CD^2 + cd^2$,
and $ca^2 = DE^2 + de^2$.

COROL. 3. Farther, because $CA^2 : ca^2 :: AD \cdot DB$ or $cd^2 : DE^2$,
therefore $CA : ca :: cd : DE$.
likewise $CA : ca :: CD : de$.

PROPOSITION XV.

If from any Point in the Curve there be drawn an Ordinate, and a Perpendicular to the Curve, or to the Tangent at that Point;

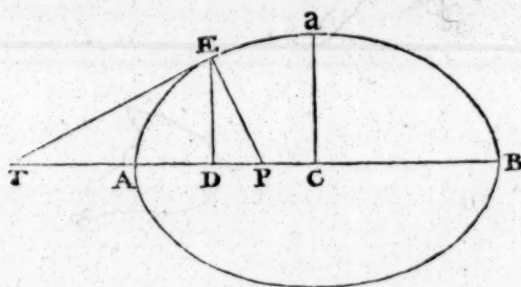
The Distance on the Transverse, between the Center and Ordinate, CD :

Will be to the Distance PD ::

As the Square of the Transverse Axis:

To the Square of the Conjugate.

That is, $CA^2 : ca^2 :: DC : DP$.



For, by prop. 2, $CA^2 : ca^2 :: AD \cdot DB : DE^2$,

But, by rt. angled $\triangle$ s, the rect. $TD \cdot DP = DE^2$;

and, by cor. I prop. 14, $CD \cdot DT = AD \cdot DB$;

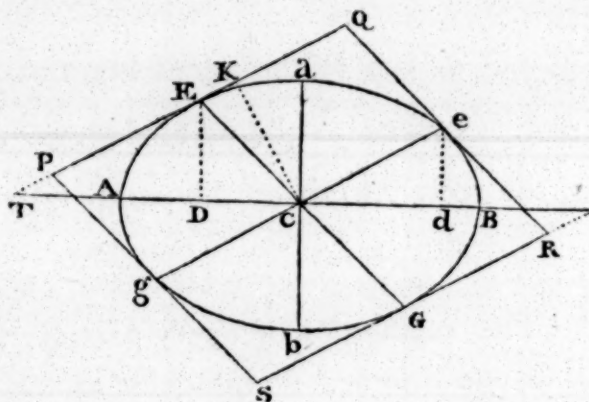
therefore $CA^2 : ca^2 :: TD \cdot DC : TD \cdot DP$,

or $AC^2 : Ca^2 :: DC : DP$. Q.E.D.

PROPOSITION XVI.

All the Parallelograms circumscribed about an Ellipse are equal to one another, and each equal to the Rectangle of the two Axes.

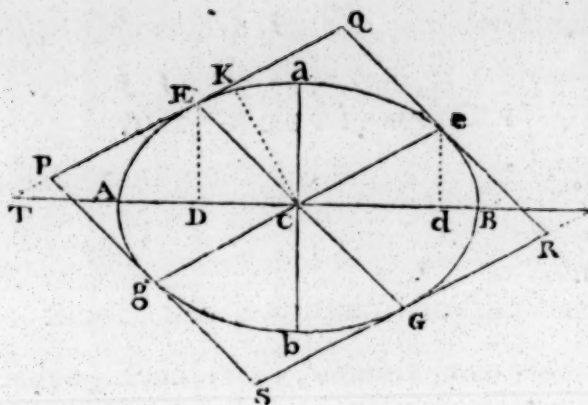
That is, the Parallelogram PQRS = the rectangle AB.ab.



Let EG, eg be two conjugate diameters parallel to the sides of the parallelogram, and dividing it into four lesser and equal parallelograms. Also draw the ordinates DE, de, and CK perpendicular to PQ.

D 2

Then,



Then, by prop. 8, $CT : CA :: CA : CD$;
 and, by cor. 3 prop. 14, $ca : de :: CA : CD$;
 theref. by equality, $CT : CA :: ca : de$.
 And by sim. tri. $CT : CK :: ce : de$,
 theref. by equality $CK : CA :: ca : ce$,
 and the rect. $CK \cdot ce = \text{rect. } CA \cdot ca$.

But the rect. $CK \cdot ce = \text{the parallelogram } CEQe$;
 theref. the rect. $CA \cdot ca = \text{the parallelogram } CEQe$;
 and, by doubling, the rect. $AB \cdot ab = \text{the paral. } PQRS$. Q.E.D.

COROL. I. The rectangles of every pair of conjugate diameters, are to one another reciprocally as the sines of their included angles. For the areas of their parallelograms, which are all equal among themselves, are equal to the rectangles of the sides, or conjugate diameters, multiplied by the sines of their contained angles, the radius being 1. That is, the rectangle of every two conjugate diameters, drawn into the sine of their contained angle, is equal to the same constant quantity. And therefore the rectangle of the diameters is inversely as the sine of their contained angle.

COROL.

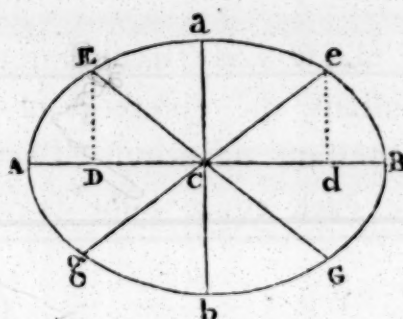
COROL. 2. As it is proved in this proposition that every circumscribing parallelogram of an ellipse is a constant quantity, so it may hence be shewn that each of the spaces $EAGP$, $EaeQ$, $GBER$, $GbgS$, between the curve and the tangents, is equal to a constant quantity. For, since every diameter bisects the ellipse, the conjugate diameters EG , eg divide the ellipse into four equal sectors $CEAg$, $CEae$, $CGBE$, $CGBg$; but the same conjugate diameters divide also the whole tangential parallelogram $PQRS$ into four equal parts, or small parallelograms $CEPg$, $CEQe$, $CGRe$, $CGSg$; and therefore the differences between these small parallelograms and the sectors, which are the said external spaces, must be all equal among themselves.

And as the ellipse and circumscribing parallelogram both remain constant, the difference of their fourth parts will also be a constant quantity. That is, the said external parts are each equal to the same constant quantity.

PROPOSITION XVII.

The Sum of the Squares of every Pair of Conjugate Diameters, is equal to the same constant Quantity, namely the Sum of the Squares of the two Axes.

That is, $AB^2 + ab^2 = EG^2 + eg^2$, where EG, eg are any conjugate diameters.



For draw the ordinates ED, ed .

Then, by cor. 2 prop. 14, $CA^2 = CD^2 + cd^2$,

and $ca^2 = DE^2 + de^2$;

therefore the sum $CA^2 + ca^2 = CD^2 + DE^2 + cd^2 + de^2$.

But, by rt. $\angle ed \triangle s$, $CE^2 = CD^2 + DE^2$,

and $ce^2 = cd^2 + de^2$;

therefore $CE^2 + ce^2 = CD^2 + DE^2 + cd^2 + de^2$.

consequently $CA^2 + ca^2 = CE^2 + ce^2$;

Or, by doubling, $AB^2 + ab^2 = EG^2 + eg^2$.

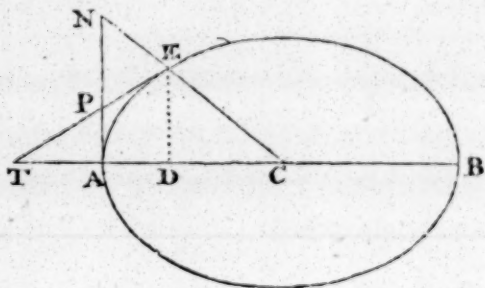
Q.E.D.

PRO-

PROPOSITION XVIII.

If there be Two Tangents drawn, the One to the Extremity of the Transverse, and the other to the Extremity of any other Diameter, each meeting the other's Diameter produced; the two Tangential Triangles so formed, will be equal.

That is, the triangle $CET =$ the triangle CAN .



For, draw the ordinate DE . Then

By sim. triangles $CD : CA :: CE : CN$;

but, by prop. 8, $CD : CA :: CA : CT$;

theref. by equal. $CA : CT :: CE : CN$.

The two triangles CET , CAN have then the angle c common, and the sides about that angle reciprocally proportional; therefore those triangles are equal.

Namely the $\triangle CET = \triangle CAN$.

Q.E.D.

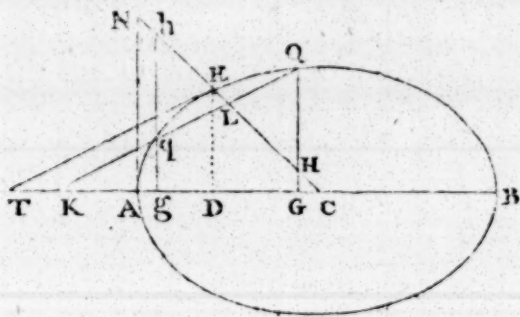
COROL. 1. From each of the equal tri. CET , CAN ,
take the common space $CAPE$,
and there remains the external $\triangle PAT = \triangle PNE$.

COROL. 2. Also from the equal triangles CET , CAN ,
take the common triangle CED ,
and there remains the $\triangle TED =$ trapez. $ANED$.

PROPOSITION XIX.

The same being supposed as in the last Proposition;
then any Lines KQ , GQ , drawn parallel to the two
Tangents, shall also cut off equal Spaces.

That is, the Triangle $KQG = \text{Trapez. } ANHG$.
and the Triangle $kqg = \text{Trapez. } ANhg$.



For draw the ordinate DE . Then

The three sim. triangles CAN , CDE , CGH ,
are to each other as CA^2 , CD^2 , CG^2 ;

theref. by div. the trap. $ANED : \text{trap. } ANHG :: CA^2 - CD^2 : CA^2 - CG^2$.

But, by prop. I, $DE^2 : GQ^2 :: CA^2 - CD^2 : CA^2 - CG^2$.

theref. by equ. trap. $ANED : \text{trap. } ANHG :: DE^2 : GQ^2$.

But, by sim. Δ s, tri. $TED : \text{tri. } KQG :: DE^2 : GQ^2$;

theref. by equal. $ANED : TED :: ANHG : KQG$.

But, by cor. 2 prop. 18, the trap. $ANED = \Delta TED$;

and therefore the trap. $ANHG = \Delta KQG$.

In like manner the trap. $ANhg = \Delta Kqg$. Q.E.D.

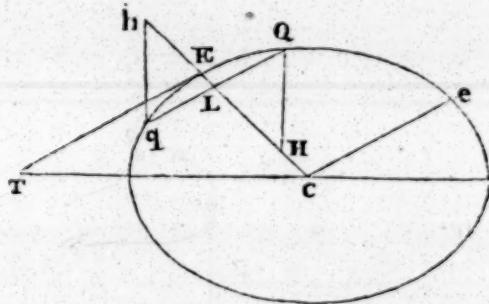
COROL. I. The three spaces $ANHG$, $TEHG$, KQG are all equal.

COROL.

PROPOSITION XX.

Any Diameter bisects all its Double Ordinates, or the Lines drawn Parallel to the Tangent at its Vertex, or to its Conjugate Diameter.

That is, if qg be parallel to the Tangent TE , or to ce , then shall $LQ = Lq$.



For draw QH , qh perpendicular to the transverse.

Then by cor. 3. prop. 18, 19, the $\triangle LQH = \triangle Lqh$;

but these triangles are also equiangular;

consequently their like sides are equal,

and therefore $LQ = Lq$.

Q.E.D.

COROL.

COROL. Any diameter divides the ellipse into two equal parts.

For, the ordinates on each side being equal to each other, and equal in number; all the ordinates, or the area, on one side of the diameter, is equal to all the ordinates, or the area, on the other side of it.

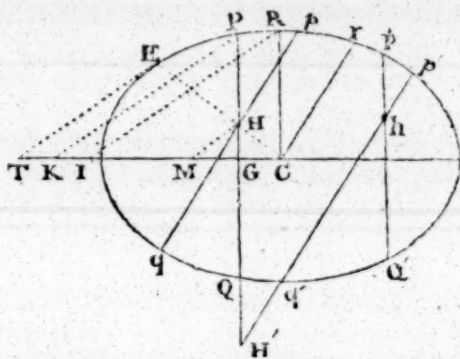
COROL. 1. The squares of the ordinates to any diameter, are to one another as the rectangles of their respective abscisses, or as the difference of the squares of the semi-diameter and of the distance between the ordinate and center. For they are all in the same ratio of CE^2 to ce^2 .

COROL. 2. The above being the same property as that belonging to the two axes, all the other properties before laid down, for the axes, may be understood of any two conjugate diameters whatever, using only the oblique ordinates of these diameters instead of the perpendicular ordinates of the axes; namely, all the properties in propositions 7, 8, 9, 12, 13, 14, 18 and 19.

PROPOSITION XXII.

If any Two Lines, that any where intersect each other,
 meet the Curve each in Two Points ; then
 The Rectangle of the Segments of the one :
 Is to the Rectangle of the Segments of the other ::
 As the Square of the Diam. Parallel to the former :
 To the Square of the Diam. Parallel to the latter.

That is, if CR and cr be Parallel to any two Lines PHQ , phq ;
 then shall $CR^2 : cr^2 :: PH \cdot HQ : ph \cdot hq$.



For draw the diameter CE , and the tangent TE and its parallels PK , RI , MH , meeting the conjugate of the diameter CR in the points T , K , I , M . Then, because similar triangles are as the squares of their like sides, we have,

by sim. triangles, $CR^2 : GP^2 :: \triangle CRI : \triangle GPK$,
 and $CR^2 : GH^2 :: \triangle CRI : \triangle GHM$;
 theref. by division, $CR^2 : GP^2 - GH^2 :: CRI : KPHM$.
 Again, by sim. tri. $CE^2 : CH^2 :: \triangle CTE : \triangle CMH$;
 and by division, $CE^2 : CE^2 - CH^2 :: \triangle CTE : TEHM$.
 But,

But, by cor. 5 prop. 19, the $\Delta CTE = \Delta CIR$,
and by cor. 1 prop. 19, $TEHG = KPHG$, or $TEHM = KPHM$;
theref. by equ. $CE^2 : CE^2 - CH^2 :: CR^2 : GP^2 - GH^2$ or $PH \cdot HQ$.
In like manner $CE^2 : CE^2 - CH^2 :: CR^2 : PH \cdot HQ$.
Theref. by equal $CR^2 : cr^2 :: PH \cdot HQ : ph \cdot hq$. Q.E.D.

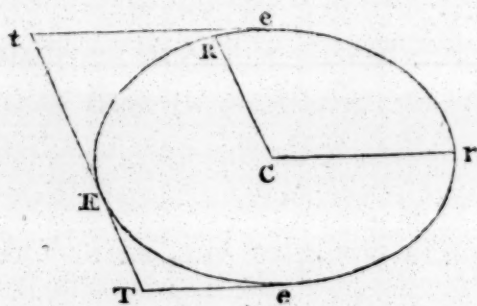
COROL. I. In like manner, if any other line $p'h'q'$, parallel to cr or to pq , meet PHQ ; since the rectangles $PH'Q$, $p'h'q'$ are also in the same ratio of CR^2 to cr^2 ; theref. the rect. $PHQ : p'hq :: PH'Q : p'h'q'$.

Also, if another line $p'hq'$ be drawn parallel to pq or cr ; because the rectangles $p'hq'$, $p'hq'$ are still in the same ratio, therefore, in general,

the rectangle $PHQ : phq :: P'hQ' : p'hq'$.

That is, the rectangles of the parts of two parallel lines, are to one another, as the rectangles of the parts of two other parallel lines, any where intersecting the former.

COROL. 2. And when any of the lines only touch the curve, instead of cutting it, the rectangles of such become squares, and the general property still attends them.



That is, $\text{CR}^2 : \text{cr}^2 :: \text{TE}^2 : \text{Te}^2$,
or $\text{CR} : \text{cr} :: \text{TE} : \text{Te}$.
and $\text{CR} : \text{cr} :: \text{te} : \text{te}$.

COROL. 3. And hence $TE : Te :: te : te$.

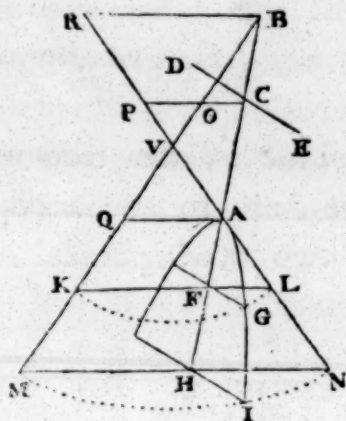
OF THE HYPERBOLA.

PROPOSITION I.

The Squares of the Ordinates of the Axis are to each other as the Rectangles of their Abscisses.

Let AVB be a plane passing through the vertex and axis of the opposite cones; $AGIH$ another section of them perpendicular to the plane of the former; AB the axis of the hyperbolic sections; and FG, HI ordinates perpendicular to it. Then

$$FG^2 : HI^2 :: AF \cdot FB : AH \cdot HB.$$



For, through the ordinates FG, HI draw the circular sections KGL, MIN parallel to the base of the cone, having KL, MN for their diameters, to which FG, HI are ordinates, as well as to the axis of the hyperbola.

Now, by the similar triangles AFL, AHN , and BFK, BHM , we have $AF : AH :: FL : HN$,

and $FB : HB :: KF : MH$;

hence, taking the rectangles of the corresponding terms, we have the rect. $AF \cdot FB : AH \cdot HB :: KF \cdot FL : MH \cdot HN$.

But, by the nature of the circle, $KF \cdot FL = FG^2$, and $MH \cdot HN = HI^2$; Therefore the rect. $AF \cdot FB : AH \cdot HB :: FG^2 : HI^2$. Q.E.D.

COROL.

COROL. 1. All the parallel sections are similar figures, or have their two axes in the same proportion; that is, $AB : ab :: DE : de$.

For, by sim. triang. $AB : ab :: AQ : aq$,
and $AB : ab :: RB : rb$;

Theref. by comp. $AB^2 : ab^2 :: AQ \cdot RB : aq \cdot rb$.

But $AQ \cdot RB = DE^2$, and $aq \cdot rb = de^2$;

Therefore $AB^2 : ab^2 :: DE^2 : de^2$,

or $AB : ab :: DE : de$.

COROL. 2. Hence also, as the property is the same for the ordinates on both sides of the diameter, it follows, that

1st. At equal distances from the center, or from the vertices, the ordinates on both sides are equal, or that the double ordinates are bisected by the axis; and that the whole figure, made up of all the double ordinates, is also bisected by the axis.

2d. The two foci are equally distant from the center, or from either vertex.

COROL. 3. When the angle, which the plane of the section makes with the base of the cone, decreases till it become equal to the angle made by the side of the cone and the base, or till the section be parallel to the opposite side of the base; then the axis becomes infinitely long, and the hyperbola degenerates into a parabola; and because then the infinites FB and HB are in a ratio of equality, the general property,

namely $AF \cdot FB : AH \cdot HB :: FG^2 : HI^2$,

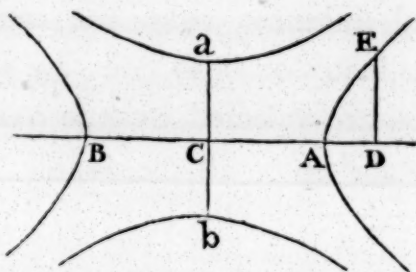
becomes $AF : AH :: FG^2 : HI^2$,

or, in the parabola, the abscissas are to each other, as the squares of their ordinates.

PROPOSITION II.

As the Square of the Transverse Axis :
 Is to the Square of the Conjugate ::
 So is the Rectangle of the Abscisses :
 To the Square of their Ordinate.

That is, $AB^2 : ab^2$ or $AC^2 : ac^2 :: AD \cdot DB : DE^2$.



For, by prop. I. $AC \cdot CB : AD \cdot DB :: ca^2 : DE^2$;
 But, if c be the center, then $AC \cdot CB = AC^2$, and ca is the semi-conj.
 Therefore $AC^2 : AD \cdot DB :: ac^2 : DE^2$;
 or, by permutation, $AC^2 : ac^2 :: AD \cdot DB : DE^2$;
 or, by doubling, $AB^2 : ab^2 :: AD \cdot DB : DE^2$. Q.E.D.

COROL. I. Or, because the rectangle $AD \cdot DB = CA^2 - CD^2$,
 the same property is $CA^2 : ca^2 :: CD^2 - CA^2 : DE^2$,
 or $AB^2 : ab^2 :: CD^2 - CA^2 : DE^2$.

COROL. 2. Or, by div. $AB : \frac{ab^2}{AB} :: CD^2 - CA^2 : DE^2$,
 that is, $AB : p :: AD \cdot DB$ or $CD^2 - CA^2 : DE^2$;
 where p is the parameter $\frac{ab^2}{AB}$ by the definition of it.

That is, As the transverse,
 Is to its parameter,
 So is the rectangle of the abscisses,
 To the square of their ordinate.

COROL. 3. When the axis AB is infinitely long, the curve becomes a parabola, and the infinites AB, DB are then in a ratio of equality; and then the last property,

namely $AB : p :: AD \cdot DB : DE^2$;

or $AB \cdot DE : AD \cdot DB :: p : DE$,

becomes $DE : AD :: p : DE$,

or $AD : DE :: DE : p$.

That is, in the parabola, the parameter is a third proportional to any absciss and its ordinate.

PROPOSITION III.

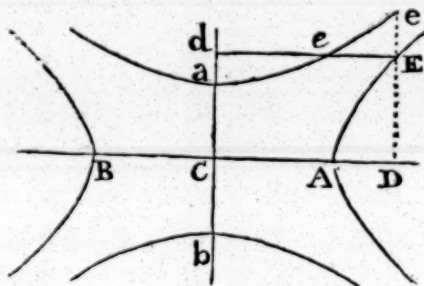
As the Square of the Conjugate Axis:

To the Square of the Transverse Axis ::

So is the Sum of the Squares of the Semi-conjugate,
and Distance of the Center from any Ordinate of this
Axis:

To the Square of that Ordinate.

That is, $ca^2 : CA^2 :: ca^2 + cd^2 : de^2$.



For draw the ordinate ED to the transverse AB.

Then, by prop. I. $CA^2 : ca^2 :: CD^2 - CA^2 : DE^2$.

But $CD^2 = DE^2$, and $DE^2 = cd^2$,

therefore $CA^2 : ca^2 :: de^2 - CA^2 : cd^2$,

or by alternation, $CA^2 : de^2 - CA^2 :: ca^2 : cd^2$,

and by composition, $CA^2 : de^2 :: ca^2 : ca^2 + cd^2$,

and by alter. & inverf. $ca^2 : CA^2 :: ca^2 + cd^2 : de^2$.

In like manner $CA^2 : ca^2 :: CA^2 + CD^2 : DE^2$. Q.E.D.

COROL.

COROL. By the last prop. $CA^2 : ca^2 :: CD^2 - CA^2 : DE^2$,

and by this prop. $CA^2 : ca^2 :: CD^2 + CA^2 : DE^2$,

therefore $DE^2 : de^2 :: CD^2 - CA^2 : CD^2 + CA^2$.

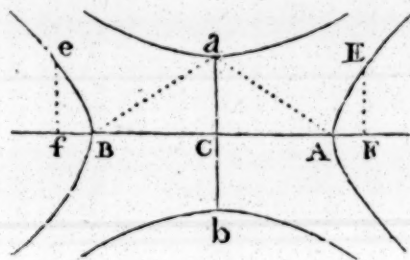
In like manner $de^2 : dE^2 :: cd^2 - ca^2 : cd^2 + ca^2$.

PROPOSITION IV.

The Square of the Distance of the Focus from the Center, is equal to the Sum of the Squares of the Semi-axes.

Or, the Square of the Distance between the Foci, is equal to the Sum of the Squares of the two Axes.

$$\begin{aligned}\text{That is, } CF^2 &= CA^2 + ca^2, \\ \text{or } Ff^2 &= AB^2 + ab^2.\end{aligned}$$



For, to the focus F draw the ordinate FE ; which, by the definition, will be the semi-parameter. Then by the nature of the curve $CA^2 : ca^2 :: CF^2 - CA^2 : FE^2$; and by the def. of the para. $CA^2 : ca^2 :: ca^2 : FE^2$;

therefore

$$ca^2 = CF^2 - CA^2;$$

and by addit.

$$CF^2 = CA^2 + ca^2;$$

or, by doubling,

$$Ff^2 = AB^2 + ab^2. \quad \text{Q.E.D.}$$

COROL. I. The two semi-axes, and the focal distance from the center, are the sides of a right angled triangle CAa ; and the distance Aa is $= CF$ the focal distance.

For

For, as above, $CA^2 + ca^2 = CF^2$,
 and by right angled Δ s, $ca^2 + ca^2 = Aa^2$,
 therefore $CF = Aa$, and $Ff = Aa + Ba$.

COROL. 2. The conjugate semi-axis ca is a mean proportional between AF , FB , or between Af , fB , the distances of either focus from the two vertices.

For $ca^2 = CF^2 - CA^2 = CF + CA \cdot CF - CA = AF \cdot FB$.

COROL. 3. The same rectangle $AF \cdot FB$ of the focal distances from either vertex, is also equal to the rectangle $AC \cdot FE$ under the semi-transverse and its semi-parameter; since this last is equal to the square of the semi-conjugate by the definition of the parameter.

Or $AF : FE :: AC : FB$.

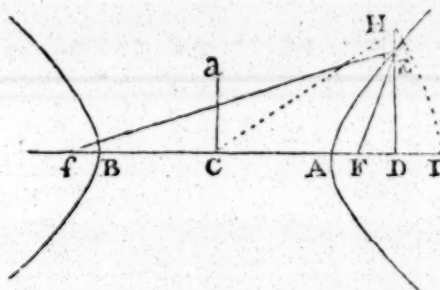
PROPOSITION V.

The Difference between the Semi-transverse and a Line drawn from the Focus to any Point in the Curve, is equal to a Fourth Proportional to the Semi-transverse, the Distance from the Center to the Focus, and the Distance from the Center to the Ordinate belonging to that Point of the Curve.

That is, $AC - FE = CI$, or $FE = AI$;

and $FE - AC = CI$, or $FE = BI$.

Where $CA : CF :: CD : CI$ the 4th proportional to CA , CF , CD .



For, by right angled triangles, $FE^2 = FD^2 + DE^2$.

Now draw AG parallel and equal to ca the semi-conjugate ; and join CG meeting the ordinate DE produced in H .

Then, by prop. 2, $CA^2 : AG^2 :: CD^2 - CA^2 : DE^2$;
 and, by sim. Δ s, $CA^2 : AG^2 :: CD^2 - CA^2 : DH^2 - AG^2$;
 consequently $DE^2 = DH^2 - AG^2 = DH^2 - ca^2$.
 Also $FD = CF \curvearrowright CD$, and $FD^2 = CF^2 - 2CF \cdot CD + CD^2$;
 therefore $FE^2 = CF^2 - ca^2 - 2CF \cdot CD + CD^2 + DH^2$.

But

But by prop. 4. $CF^2 + ca^2 = CA^2$

and, by supposition, $2CF \cdot CD = 2CA \cdot CI$;

theref. $FE^2 = CA^2 - 2CA \cdot CI + CD^2 + DH^2$.

But, by supposition $CA^2 : CD^2 :: CF^2$ or $CA^2 + AG^2 : CI^2$;

and, by sim. Δs , $CA^2 : CD^2 :: CA^2 + AG^2 : CD^2 + DH^2$;

therefore $CI^2 = CD^2 + DH^2 = CH^2$;

consequently $FE^2 = CA^2 - 2CA \cdot CI + CI^2$.

And the root or side of this square is $FE = CI - CA = AI$.

In the same manner is found $FE = CI + CA = BI$. Q.E.D.

COROL. 1. Hence $CH = CI$ is a 4th proportional to CA, CF, CD .

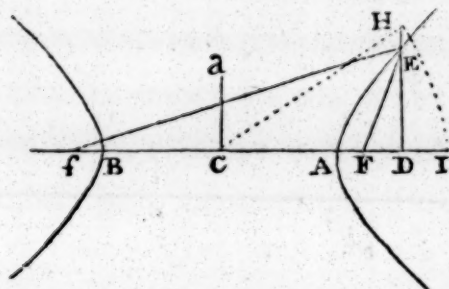
COROL. 2. And $FE + FE = 2CH$ or $2CI$; or FE, CH, FE are in continued arithmetical progression, the common difference being CA the semi-transverse.

COROL. 3. From the demonstration it appears that $DE^2 = DH^2 - AG^2 = DH^2 - ca^2$. Consequently DH is every where greater than DE ; and so the asymptote CGH never meets the curve, though they be ever so far produced: but DH and DE approach nearer and nearer to a ratio of equality as they recede farther from the vertex, till at an infinite distance they become equal, and the asymptote is a tangent to the curve at an infinite distance from the vertex.

PROPOSITION VI.

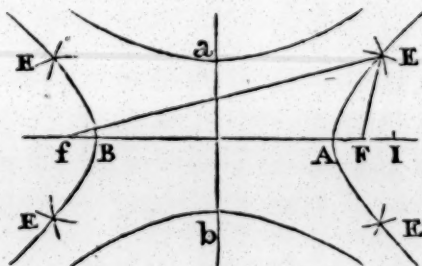
The Difference of two Lines drawn from the Foci, to meet in any Point of the Curve, is equal to the Transverse Axis.

That is, $fE - FE = AB$.



For, by the last prop. $fE = CI - CA = AI$,
and, by the same, $fE = CI + CA = BI$;
theref. by subtraction, $fE - FE = AB$.

COROL. Hence is derived the common method of describing the curve mechanically by points, thus.



In

In the transverse AB , produced, take the foci F, f , and any point i . Then with the radii AI, BI , and centers F, f , describe arcs intersecting in E , which will be a point in the curve. In like manner, assuming other points i , as many other points will be found in the curve.

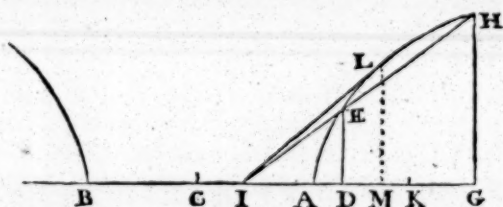
Then with a steady hand, draw the curve line through all the points of intersection E .

In the same manner are constructed the other two hyperbolas, using the axis ab instead of AB .

PROPOSITION VII.

If from any Point I in the Axis produced, a Line IEH be drawn cutting the curve in Two Points; and from those Two Points be drawn the Perpendicular Ordinates DE , GH ; and if K be the Middle of DG , and C the Center or the Middle of AB : Then shall CK be to CI as the Rectangle of AD and AG to the Square of AI .

That is, $CK : CI :: AD \cdot AG : AI^2$.



For, by prop. I. $AD \cdot DB : AG \cdot GB :: DE^2 : GH^2$,

and by sim. Δ s, $ID^2 : IG^2 :: DE^2 : GH^2$;

theref. by equal. $AD \cdot DB : AG \cdot GB :: ID^2 : IG^2$.

But $DB = 2CK - AG$, and $GB = 2CK - AD$,

theref. $AD \cdot 2CK - AD \cdot AG : AG \cdot 2CK - AD \cdot AG :: ID^2 : IG^2$,

and, by div. $DG \cdot 2CK : IG^2 - ID^2$ or $DG \cdot 2IK :: AD \cdot 2CK - AD \cdot AG : ID^2$.

or $2CK : 2IK :: AD \cdot 2CK - AD \cdot AG : ID^2$,

or $AD \cdot 2CK : AD \cdot 2IK :: AD \cdot 2CK - AD \cdot AG : ID^2$;

theref.

theref. by div. $CK : IK :: AD \cdot AG : AD \cdot 2IK - ID^2$,

and, by div. $CK : CI :: AD \cdot AG : ID^2 - AD \cdot ID + IA$,

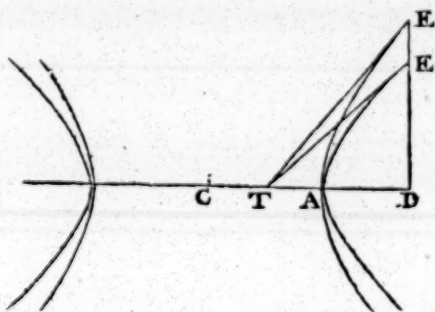
or $CK : CI :: AD \cdot AG : AI^2$. Q.E.D.

COROL. When the line IH , by revolving about the point I , comes into the position of the tangent IL , and the ordinate LM being drawn, then the points E and H meet in the point L , and the points D , K , G , coincide with the point M ; and then the property in the proposition becomes $CM : CI :: AM^2 : AI^2$.

PROPOSITION VIII.

If a Tangent and Ordinate be drawn from any Point in the Curve, meeting the Transverse Axis; the Semi-transverse will be a Mean Proportional between the Distances of the said Two Intersections from the Center.

That is, CA is a mean proportion between CD and CT ; or CD, CA, CT are continued proportionals.



For, by cor. prop. 7, $CD : CT :: AD^2 : AT^2$,
 that is, $CD : CT :: \overline{CD - CA}^2 : \overline{CA - CT}^2$,
 or $:: CD^2 + CA^2 : CA^2 + CT^2$,
 and $:: CD^2 : CA^2$,
 and $:: CA^2 : CT^2$;
 therefore $CD : CA :: CA : CT$. Q.E.D.

COROL. I. Since CT is always a third proportional to CD, CA ; if the points D, A , remain constant, then will the point T be constant also; and therefore all the tangents will meet in this point T , which are drawn from E , of every

every hyperbola described on the same axis AB , where they are cut by the common ordinate DEE drawn from the point D .

COROL. 2. Hence a tangent is easily drawn to the curve, from any point, either in the curve or without it.

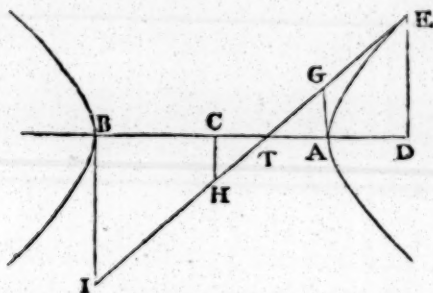
First, if the given point E be in the curve. Draw the ordinate DE of the diameter AC ; and in the diameter produced take CT a third proportional to CD , CA . Then join TE for the tangent required.

But if the point T be given any where without the curve. Join CT , in which take CD a third proportional to CT , CA ; and draw the ordinate DE . Then join TE as before.

PROPOSITION IX.

If there be any Tangent meeting Four Perpendiculars to the Axis drawn from these four Points, namely the Center, the two Extremities of the Axis, and the Point of Contact; those Four Perpendiculars will be Proportionals.

That is, $AG : DE :: CH : BI$.



For, by prop. 8, $TC : AC :: AC : DC$,
 theref. by div. $TA : AD :: TC : AC$ or CB ,
 and by comp. $TA : TD :: TC : TB$,
 and by fim. tri. $AG : DE :: CH : BI$.

Q.E.D.

COROL. Hence TA, TD, TC, TB } are also proportionals.
 and TG, TE, TH, TI }

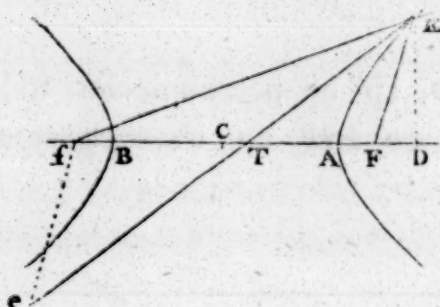
For these are as AG, DE, CH, BI , by similar triangles.

PRO-

PROPOSITION X.

If there be any Tangent, and two Lines drawn from the Foci to the Point of Contact; these two Lines will make equal Angles with the Tangent.

That is, the $\angle FET = \angle fee$.



For draw the ordinate DE, and fe parallel to FE.

By cor. I. prop. 5, $CA : CD :: CF : CA + FE$,
 and by prop. 8, $CA : CD :: CT : CA$;
 therefore $CT : CF :: CA : CA + FE$;
 and by add. and sub. $TF : Tf :: FE : 2CA + FE$ or fe by prop. 6.
 But by sim. tri. $TF : Tf :: FE : fe$;
 therefore $fe = fe$, and conseq. $\angle e = \angle fee$.
 But, because FE is parallel to fe , the $\angle e = \angle FET$;
 therefore the $\angle FET = \angle fee$. Q.E.D.

COROL. As opticians find that the angle of incidence is equal to the angle of reflection, it appears from our proposition, that rays of light issuing from the one focus, and meeting the curve in every point, will be reflected into lines drawn from the other focus. So the ray fe is reflected into FE . And this is the reason why the points F, f are called *foci*, or burning points.

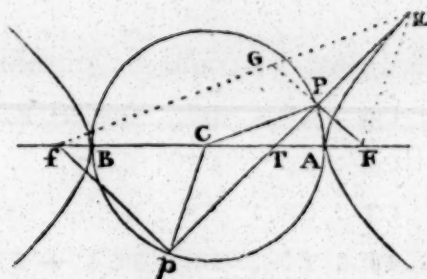
F

PRO-

PROPOSITION XI.

If a Line be drawn from either Focus, Perpendicular to a Tangent to any Point of the Curve; the Distance of their Interfection from the Center will be equal to the Semi-transverse Axis.

That is, if FP , fp be perpendicular to the Tangent TPP , then shall CP and cp be each equal to CA or CB .



For, through the point of contact E draw FE and fe meeting FP produced in G . Then, the $\angle GEP = \angle FEP$, being each equal to the $\angle fep$, and the angles at P being right, and the side PE being common, the two triangles GEP , FEP are equal in all respects, and so $GE = FE$, and $GP = FP$. Therefore, since $FP = \frac{1}{2}FG$, and $FC = \frac{1}{2}Ff$, and the angle at F common, the side CP will be $= \frac{1}{2}fg$ or $\frac{1}{2}AB$, that is $CP = CA$ or CB .

And in the same manner $cp = CA$ or CB . Q.E.D.

COROL. 1. A circle described on the transverse axis, as a diameter, will pass through the points P, p ; because all the lines CA, CP, cp, CB , being equal, will be radii of the circle.

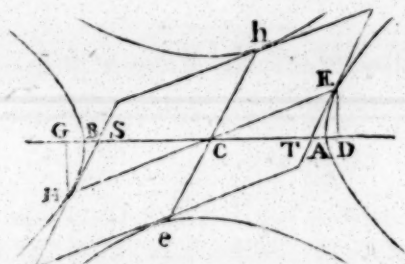
COROL. 2. CP is parallel to fE , and cp parallel to FE .

COROL. 3. If at the intersections of any tangent, with the circumscribed circle, perpendiculars to the tangent be drawn, they will meet the transverse axis in the two foci. That is, the perpendiculars PF, pf give the foci F, f .

PROPOSITION XII.

The equal Ordinates, or the Ordinates at equal Distances from the Center, on the opposite Sides and Ends of an Ellipse, have their Extremities connected by one Right Line passing through the Center, and that Line is bisected by the Center.

That is, if $CD = CG$, or the ordinate $DE = GH$; then shall $CE = CH$, and ECH will be a right line.



For, when $CD = CG$, then also is $DE = GH$ by cor. 2. prop. 1. But the $\angle D = \angle G$, being both right angles; therefore the third side $CE = CH$, and the $\angle DCE = \angle GCH$, and consequently ECH is a right line.

COROL. I. And, conversely, if ECH be a right line passing through the center; then shall it be bisected by the center, or have $CE = CH$; also DE will be $= GH$, and $CD = CG$.

COROL.

COROL. 2. Hence also, if two tangents be drawn to the two ends E, H of any diameter EH; they will be parallel to each other, and will cut the axis at equal angles, and at equal distances from the center. For, the two CD, CA being equal to the two CG, CB, the third proportionals CT, CS will be equal also; then the two sides CE, CT being equal to the two CH, CS, and the included angle ECT equal to the included angle HCS, all the other corresponding parts are equal: and so the $\angle T = \angle S$, and TE parallel to HS.

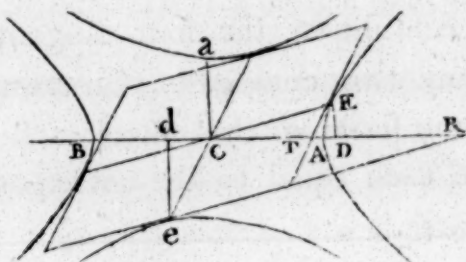
COROL. 3. And hence the four tangents, at the four extremities of any two conjugate diameters, form a parallelogram circumscribing the ellipse, and the pairs of opposite sides are each equal to the corresponding parallel conjugate diameters.

For, if the diameter eh be drawn parallel to the tangent TE or HS, it will be the conjugate to EH by the definition; and the tangents to eh will be parallel to each other, and to the diameter EH for the same reason.

PROPOSITION XIII.

If two Ordinates ED , ed be drawn from the Extremities E , e , of two Conjugate Diameters, and Tangents be drawn to the same Extremities, and meeting the Axis produced in T and R ;

Then shall CD be a mean Proportional between cd , dr ,
and cd a mean Proportional between CD , DT .



For, by prop. 8, $CD : CA :: CA : CT$,
and by the same $cd : CA :: CA : CR$;
theref. by equality $CD : cd :: CR : CT$.
But by sim. tri. $DT : cd :: CT : CR$;
theref. by equality $CD : cd :: cd : DT$.
In like manner $cd : CD :: CD : dr$. Q.E.D.

COROL. 1. Hence $CD : cd :: CR : CT$.

COROL. 2. Hence also $CD : cd :: de : DE$.
And the rect. $CD \cdot DE = cd \cdot de$, or $\triangle CDE = \triangle cde$.

COROL. 3. Also $cd^2 = CD \cdot DT$,
and $CD^2 = cd \cdot dr$.

Or cd a mean proportional between CD , DT ;
and CD a mean proportional between cd , dr .

PRO-

PROPOSITION XIV.

The same Figure being constructed as in the last Proposition, each Ordinate will divide the Axis, and the Semi-axis added to the external Part, in the same Ratio.

That is, $DA : DT :: DC : DB$,
and $dA : dR :: dc : dB$.

[See the last fig.]

For, by prop. 8, $CD : CA :: CA : CT$,

and by div. $CD : CA :: AD : AT$,

and by comp. $CD : DB :: AD : DT$,

or $DA : DT :: DC : DB$.

In like manner $dA : dR :: dc : dB$.

Q.E.D.

COROL. 1. Hence, and from cor. 3 to the last prop. we have

$$cd^2 = CD \cdot DT = AD \cdot DB = CD^2 - CA^2,$$

$$CD^2 = cd \cdot dR = Ad \cdot dB = CA^2 - cd^2.$$

COROL. 2. Hence also $CA^2 = CD^2 - cd^2$,
and $ca^2 = de^2 - DE^2$.

COROL. 3. Farther, because $CA^2 : ca^2 :: AD \cdot DB$ or $cd^2 : DE^2$,
therefore $CA : ca :: cd : DE$.
likewise $CA : ca :: CD : de$.

PROPOSITION XV.

If from any Point in the Curve there be drawn an Ordinate, and a Perpendicular to the Curve, or to the Tangent at that Point;

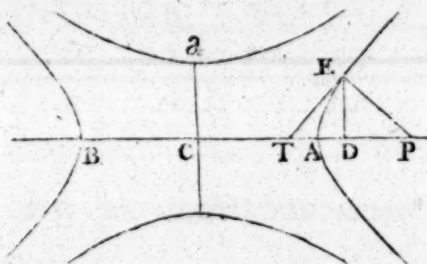
The Distance on the Transverse, between the Center and Ordinate, CD :

Will be to the Distance PD ::

As the Square of the Transverse Axis :

To the Square of the Conjugate.

That is, $CA^2 : ca^2 :: DC : DP$.



For, by prop. 2, $CA^2 : ca^2 :: AD \cdot DB : DE^2$,

But, by rt. angled Δ s, the rect. $TD \cdot DP = DE^2$;

and, by cor. I prop. 14, $CD \cdot DT = AD \cdot DB$;

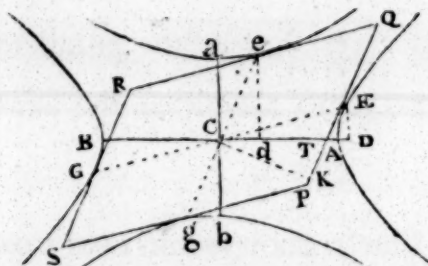
therefore $CA^2 : ca^2 :: TD \cdot DC : TD \cdot DP$,

or $CA^2 : ca^2 :: DC : DP$. Q.E.D.

PROPOSITION XVI.

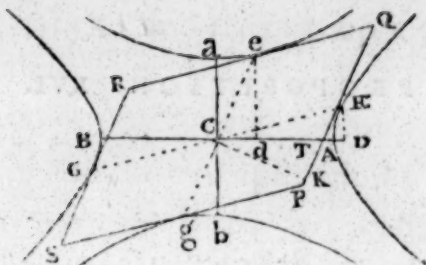
All the Parallelograms inscribed between the four Conjugate Hyperbolas are equal to one another, and each equal to the Rectangle of the two Axes.

That is, the Parallelogram PQRS = the rectangle $AB \cdot ab$.



Let EG, eg be two conjugate diameters parallel to the sides of the parallelogram, and dividing it into four lesser and equal parallelograms. Also draw the ordinates DE, de, and CK perpendicular to PQ.

Then,



Then, by prop. 8, $CT : CA :: CA : CD$;

and, by cor. 3 prop. 14, $ca : de :: CA : CD$;

theref. by equality, $CT : CA :: ca : de$.

And by sim. tri. $CT : CK :: ce : de$,

theref. by equality $CK : CA :: ca : ce$,

and the rect. $CK \cdot ce = \text{rect. } CA \cdot ca$.

But the rect. $CK \cdot ce = \text{the parallelogram } CEQE$;

theref. the rect. $CA \cdot ca = \text{the parallelogram } CEQE$;

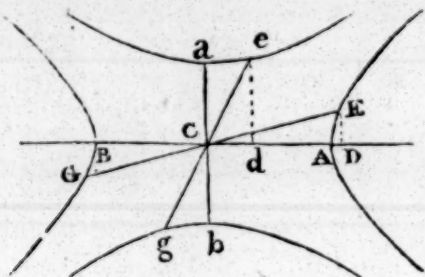
and, by doubling, the rect. $AB \cdot ab = \text{the paral. } PQRS. \quad Q.E.D.$

COROL. I. The rectangles of every pair of conjugate diameters, are to one another reciprocally as the fines of their included angles. For the areas of their parallelograms, which are all equal among themselves, are equal to the rectangles of the sides, or conjugate diameters, multiplied by the fines of their contained angles, the radius being 1. That is, the rectangle of every two conjugate diameters, drawn into the sine of their contained angle, is equal to the same constant quantity. And therefore the rectangle of the diameters is inverfely as the sine of their contained angle.

PROPOSITION XVII.

The Difference of the Squares of every Pair of Conjugate Diameters, is equal to the same constant Quantity, namely the Difference of the Squares of the two Axes.

That is, $AB^2 - ab^2 = EG^2 - eg^2$, where EG, eg are any conjugate diameters.



For draw the ordinates ED, ed .

Then, by cor. 8 prop. 9, $CA^2 = CD^2 - cd^2$,

and $ca^2 = de^2 - DE^2$;

theref. the difference $CA^2 - ca^2 = CD^2 + DE^2 - cd^2 - de^2$.

But, by rt. $\angle ed \Delta s$, $CE^2 = CD^2 + DE^2$,

and $ce^2 = cd^2 + de^2$;

therefore $CE^2 - ce^2 = CD^2 + DE^2 - cd^2 - de^2$.

consequently $CA^2 - ca^2 = CE^2 - ce^2$;

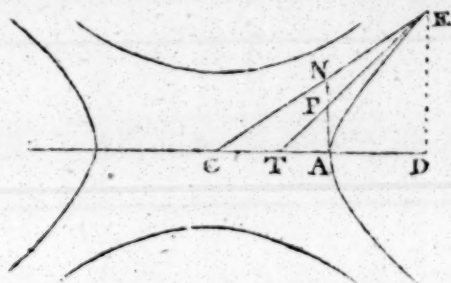
Or, by doubling, $AB^2 - ab^2 = EG^2 - eg^2$. Q.E.D.

P R O-

PROPOSITION XVIII.

If there be Two Tangents drawn, the One to the Extremity of the Transverse, and the other to the Extremity of any other Diameter, each meeting the other's Diameter produced; the two Tangential Triangles so formed, will be equal.

That is, the triangle $CET =$ the triangle CAN .



For, draw the ordinate DE . Then

By sim. triangles $CD : CA :: CE : CN$;

but, by prop. 8, $CD : CA :: CA : CT$;

theref. by equal. $CA : CT :: CE : CN$.

The two triangles CET , CAN have then the angle c common, and the sides about that angle reciprocally proportional; therefore those triangles are equal.

Namely the $\triangle CET = \triangle CAN$.

Q.E.D.

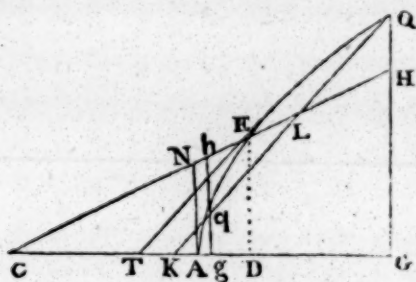
COROL. 1. Take each of the equal tri. CET, CAN,
from the common space CAPE,
and there remains the external $\triangle PAT = \triangle PNE$.

COROL. 2. Also take the equal triangles CET, CAN,
from the common triangle CED,
and there remains the $\triangle TED = \text{trapez. ANED}$.

PROPOSITION XIX.

The same being supposed as in the last Proposition;
then any Lines KQ , GQ , drawn parallel to the two
Tangents, shall also cut off equal Spaces.

That is, the Triangle $KQG = \text{Trapez. ANHG}$.
and the Triangle $kqg = \text{Trapez. ANhg}$.



For draw the ordinate DE . Then

The three sim. triangles CAN , CDE , CGH ,
are to each other as CA^2 , CD^2 , CG^2 ;

theref. by div. the trap. $ANED : \text{trap. ANHG} :: CD^2 - CA^2 : CG - CA^2$.

But, by prop. 1, $DE^2 : GQ^2 :: CD^2 - CA^2 : CG^2 - CA^2$.

theref. by equ. trap. $ANED : \text{trap. ANHG} :: DE^2 : GQ^2$.

But, by sim. Δ s, tri. $TED : \text{tri. KQG} :: DE^2 : GQ^2$;

theref. by equal. $ANED : TED :: ANHG : KQG$.

But, by cor. 2 prop. 18, the trap. $ANED = \Delta TED$;

and therefore the trap. $ANHG = \Delta KQG$.

In like manner the trap. $ANhg = \Delta kqg$. Q.E.D.

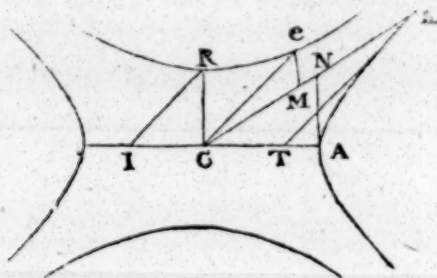
COROL. 1. The three spaces $ANHG$, $TEHG$, KQG are all equal.

COROL.

COROL. 2. From the equals $ANHG$, KQG ,
take the equals $ANhg$, Kqg ,
and there remains $ghHG = gqQG$.

COROL. 3. And from the equals $ghHG$, $gqQG$,
take the common space $gqLHG$,
and there remains the $\triangle LQH = \triangle Lqh$.

COROL. 4. Again from the equals KQG , $TEHG$,
take the common space $KLHG$,
and there remains $TELK = \triangle LQH$.



COROL. 5. And when, by the lines KQ , GH , moving
with a parallel motion, KQ comes into the position IR ,
where CR is the conjugate to CA ; then

the triangle KQG becomes the triangle IRC ,
and the space $ANHG$ becomes the triangle ANC ;
and therefore the $\triangle IRC = \triangle ANC = \triangle TEC$.

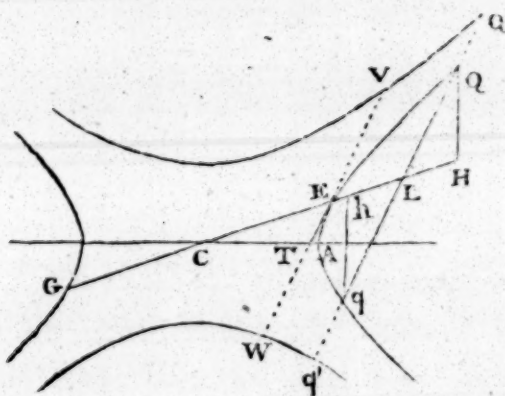
COROL. 6. Also when the lines KQ and HQ , by moving
with a parallel motion, come into the position ce , me ,
the triangle LQH becomes the triangle cem ,
and the space $TELK$ becomes the triangle TEC ;
and theref. the $\triangle cem = \triangle TEC = \triangle ANC = \triangle IRC$,

PRO-

PROPOSITION XX.

Any Diameter bisects all its Double Ordinates, or the Lines drawn Parallel to the Tangent at its Vertex, or to its Conjugate Diameter.

That is, if qq be parallel to the Tangent TE , or to ce , then shall $LQ = Lq$.



For draw QH , qh perpendicular to the transverse.
 Then by cor. 3 prop. 19, the $\triangle LQH = \triangle Lqh$;
 but these triangles are also equiangular;
 consequently their like sides are equal,
 and therefore $LQ = Lq$.

Q.E.D.

COROL.

COROL. 1. Any diameter divides the ellipse into two equal parts.

For, the ordinates on each side being equal to each other, and equal in number; all the ordinates, or the area, on one side of the diameter, is equal to all the ordinates, or the area, on the other side of it.

COROL. 2. In like manner, if the ordinate be produced to the conjugate hyperbolas at Q' , q' , it may be proved that $LQ' = Lq'$. Or if the tangent TE be produced, then $EV = EW$. And the diameter $GCEH$ bisects all lines drawn parallel to TE or Qq , and limited either by one hyperbola, or by its two conjugate hyperbolas.

COROL. 1. The squares of the ordinates to any diameter, are to one another as the rectangles of their respective abscisses, or as the difference of the squares of the semi-diameter and of the distance between the ordinate and center. For they are all in the same ratio of CE^2 to ce^2 .

COROL. 2. The above being the same property as that belonging to the two axes, all the other properties before laid down, for the axes, may be understood of any two conjugate diameters whatever, using only the oblique ordinates of these diameters instead of the perpendicular ordinates of the axes; namely, all the properties in propositions 7, 8, 9, 12, 13, 14, 18 and 19.

COROL. 3. Likewise, when the ordinates are continued to the conjugate hyperbolas at Q' , q' , the same properties still obtain, substituting only the sum for the difference of the squares of CE and CL ,

That is, $CE^2 : ce^2 :: CL^2 + CE^2 : CQ'^2$.

And so $LQ^2 : LQ'^2 :: CL^2 - CE^2 : CL^2 + CE^2$.

COROL. 4. When, by the motion of LQ parallel to itself, that line coincides with EV , the last corollary becomes

$$CE^2 : ce^2 :: 2CE^2 : EV^2,$$

$$\text{or} \quad ce^2 : EV^2 :: 1 : 2,$$

$$\text{or} \quad ce : EV :: 1 : \sqrt{2},$$

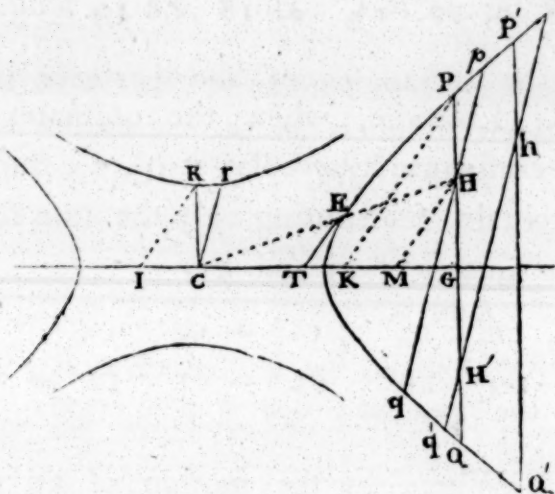
or as the side of a square to its diagonal.

That is, in all conjugate hyperbolas; and all their diameters, any diameter is to its parallel tangent, in the constant ratio of the side of a square to its diagonal.

PROPOSITION XXII.

If any Two Lines, that any where intersect each other,
meet the Curve each in Two Points; then
The Rectangle of the Segments of the one :
Is to the Rectangle of the Segments of the other ::
As the Square of the Diam. Parallel to the former :
To the Square of the Diam. Parallel to the latter.

That is, if CR and cr be Parallel to any two Lines PHQ , phq ;
then shall $CR^2 : cr^2 :: PH \cdot HQ : ph \cdot hq$.



For draw the diameter CH , and the tangent TE and its parallels PK , RI , MH , meeting the conjugate of the diameter CR in the points T , K , I , M . Then, because similar triangles are as the squares of their like sides, we have,

by sim. triangles, $CR^2 : GP^2 :: \triangle CRI : \triangle GPK$,
and $CR^2 : GH^2 :: \triangle CRI : \triangle GHM$;
theref. by division, $CR^2 : GP^2 - GH^2 :: CRI : KPHM$.

Again, by sim. tri. $CE^2 : CH^2 :: \triangle CTE : \triangle CMH$;
and by division, $CE^2 : CH^2 - CE^2 :: \triangle CTE : TEHM$.

But,

But, by cor. 5 prop. 19, the $\triangle CTE = \triangle CIR$,
 and by cor. 1 prop. 19, $TEHG = KPHG$, or $TEHM = KPHM$;
 theref. by equ. $CE^2 : CH^2 - CE^2 :: CR^2 : GP^2 - GH^2$ or $PH \cdot HQ$.
 In like manner $CE^2 : CH^2 - CE^2 :: cr^2 : PH \cdot Hq$.
 Theref. by equal $CR^2 : cr^2 :: PH \cdot HQ : PH \cdot Hq$. Q.E.D.

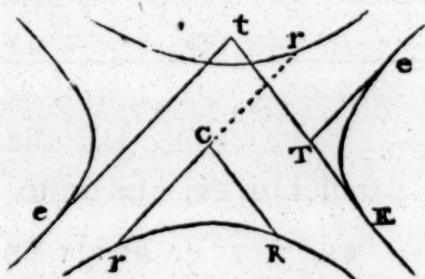
COROL. 1. In like manner, if any other line $p'h'q'$,
 parallel to cr or to pq , meet PHQ ; since the rectangles
 $PH'Q$, $p'h'q'$ are also in the same ratio of CR^2 to cr^2 ;
 theref. the rect. $PHQ : PHq :: PH'Q : p'h'q'$.

Also, if another line $p'hq'$ be drawn parallel to PQ
 or CR ; because the rectangles $p'hq'$, $p'hq'$ are still in the
 same ratio, therefore, in general,

the rectangle $PHQ : PHq :: p'hq' : p'hq'$.

That is, the rectangles of the parts of two parallel
 lines, are to one another, as the rectangles of the parts
 of two other parallel lines, any where intersecting the
 former.

COROL. 2. And when any of the lines only touch the
 curve, instead of cutting it, the rectangles of such become
 squares, and the general property still attends them.



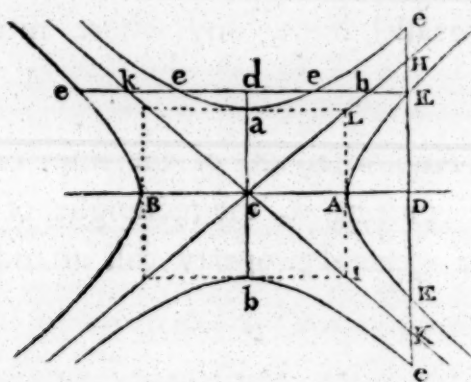
That is, $CR^2 : cr^2 :: TE^2 : Te^2$,
 or $CR : cr :: TE : Te$.
 and $CR : cr :: TE : te$.

COROL. 3. And hence $TE : Te :: TE : te$.

PROPOSITION XXIII.

If a Line be drawn through any Point of the Curves, Parallel to either of the Axes, and terminated at the Asymptotes; the Rectangle of its Segments, measured from that Point, will be equal to the Square of the Semi-axis to which it is parallel.

That is, the Rect. HEK or HEK = ca^2 ,
and the Rect. hek or hek = CA^2 .



For draw AL parallel to ca, and aL to CA. Then
by the parallels, $CA^2 : ca^2$ or $AL^2 :: CD^2 : DH^2$;
and, by prop. 2, $CA^2 : ca^2 :: CD^2 - CA^2 : DE^2$;
theref. by subtr. $CA^2 : ca^2 :: CA^2 : DH^2 - DE^2$ or HEK.
But the antecedents CA^2 , CA^2 are equal,
theref. the consequents ca^2 , HEK must also be equal.

In like manner we have, again
 by the parallels, $CA^2 : ca^2$ or $AL^2 :: CD^2 : DH^2$;
 and, by prop. 3, $CA^2 : ca^2 :: CD^2 + CA^2 : De^2$;
 theref. by subtr. $CA^2 : ca^2 :: CA^2 : De^2 - DH^2$ or HEK .
 But the antecedents CA^2 , ca^2 are the same,
 theref. the consequents ca^2 , HEK must be equal.
 In like manner, by changing the axes, is hek or $hek = CA^2$. Q.E.D.

COROL. 1. Because the rect. $HEK =$ the rect. HEK .
 therefore $EH : eH :: eK : EK$.

And consequently HE is always greater than he .

COROL. 2. The rectangle $hek =$ the rect. HEK .
 For, by sim. tri. $eh : EH :: ek : EK$.

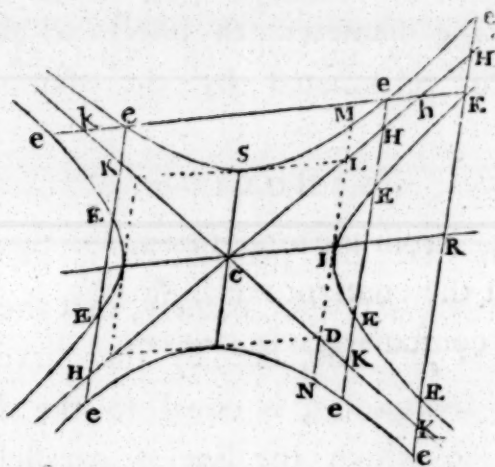
SCHOLIUM.

It is evident that this proposition is general for any line oblique to the axis also, namely, that the rectangle of the segments of any line, cut by the curve, and terminated by the asymptotes, is equal to the square of the semi-diameter to which the line is parallel. Since the demonstration is drawn from properties that are common to all diameters.

PROPOSITION XXIV.

All the Rectangles are equal which are made of the Segments of any Parallel Lines cut by the Curve, and limited by the Asymptotes.

That is, the Rect. $HEK = \text{Rect. } hek$.
and the Rect. $hek = hek$.



For each of the rectangles HEK or hek is equal to the square of the parallel semi-diameter cs ; and each of the rectangles hek or hek is equal to the square of the parallel semi-diameter ci . And therefore the rectangles of the segments of all parallel lines are equal to one another.

Q.E.D.

COROL.

COROL. 1. The rectangle HEK being constantly the same, whether the point E is taken on the one side or the other of the point of contact I of the tangent parallel to HK, it follows that the parts HE, KE, of any line HK, are equal.

And because the rectangle HEK is constant, whether the point e is taken in the one or the other of the opposite hyperbolas, it follows that the parts He, Ke are also equal.

COROL. 2. And when HK comes into the position of the tangent DIL, the last corollary becomes $IL = ID$, and $IM = IN$, and $LM = DN$.

Hence also the diameter CIR bisects all the parallels to DL which are terminated by the asymptote, namely $RH = RK$.

COROL. 3. From the proposition, and the last corollary, it follows that the constant rectangle HEK or EHE is $= IL^2$. And the equal constant rect. HEK or EHE $= MLN$ or $IM^2 - IL^2$.

COROL. 4. And hence IL = the parallel semi-diameter cs.

For the rect. EHE $= IL^2$,
 and the equal rect. eHe $= IM^2 - IL^2$,
 theref. $IL^2 = IM^2 - IL^2$, or $IM^2 = 2IL^2$;
 but, by cor. 4 prop. 21, $IM^2 = 2cs^2$,
 and therefore $IL = cs$.

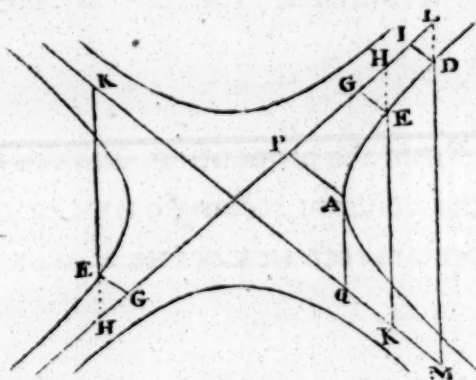
And so the asymptotes passes through the opposite angles of all the inscribed parallelograms.

PRO.

PROPOSITION XXV.

The Rectangle of any two Lines drawn from any Point in the Curve, Parallel to two given Lines, and Limited by the Asymptotes, is a Constant Quantity.

That is, if AP, EG, DI be Parallels,
as also AQ, EK, DM Parallels,
then shall the rect. $PAQ = \text{rect. } GEK = \text{rect. } IDM.$



For, produce KE, MD to the other asymptote at $H, L.$

Then, by the parallels, $HE : GE :: LD : ID;$

but, $EK : EK :: DM : DM;$

theref. the rectangle $HEK : GEK :: LDM : IDM.$

But, by the last prop. the rect. $HEK = LDM;$

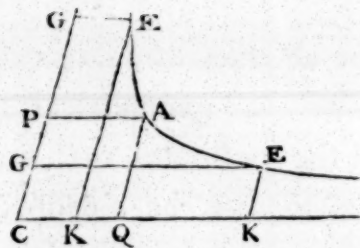
and therefore the rect. $GEK = IDM = PAQ. \quad Q.E.D.$

PRO-

PROPOSITION XXVI.

All the Parallelograms are equal which are formed between the Asymptotes and Curve, by Lines Parallel to the Asymptotes.

That is, the paral. $CGEK = CPAQ$.



For, by prop. 25, the rect. GEK or $CGE = \text{rect. } PAQ \text{ or } CPA$.

But the parallelograms $CGEK$, $CPAQ$ being equiangular,
are as the rectangles CGE , CPA .

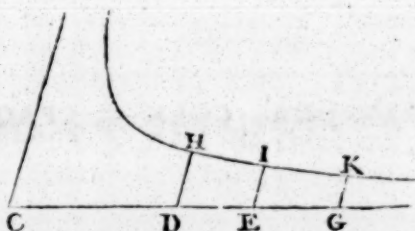
And therefore the parallelogram $GK = PQ$.

That is, all the inscribed parallelograms are equal to one
another.

Q.E.D.

COROL.

COROL. 1. Because the rectangle GEK or CGE is constant, therefore GE is reciprocally as CG, or $CG : CP :: PA : GE$. And hence the asymptote continually approaches towards the curve, but never meets it: for GE decreases continually as CG increases; and it is always of *some* magnitude, except when CG is supposed to be infinitely great, for then GE is infinitely small, or nothing. So that the asymptote CG may be considered as a tangent to the curve at a point infinitely distant from C.



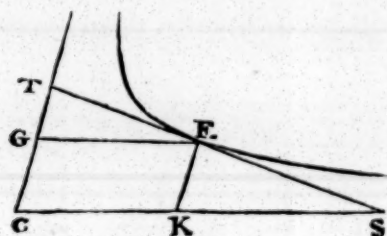
COROL. 2. If the abscisses CD, CE, CG, &c, taken on the one asymptote, be in geometrical progression increasing; then shall the ordinates DH, EI, GK, &c, parallel to the other asymptote, be a decreasing geometrical progression, having the same ratio. For, all the rectangles CDH, CEI, CGK, &c, being equal, the ordinates DH, EI, GK, &c, are reciprocally as the abscisses CD, CE, CG, &c, which are geometricals. And the reciprocals of geometricals are also geometricals, and in the same ratio.

COROL. 3. But if the abscisses CD, CE, CG, &c, be arithmeticals, or in arithmetical progression, then shall the ordinates DH, EI, GK, &c, be harmonicals, or in harmonical progression. For the reciprocals of arithmeticals, are harmonicals.

PROPOSITION XXVII.

Every Inscribed Triangle, formed by Any Tangent and the two Intercepted Parts of the Asymptotes, is equal to a Constant Quantity; namely Double the Inscribed Parallelogram.

That is, the triangle $cts = 2$ paral. gk .



For, since the tangent ts is bisected by the point of contact e , and ek is parallel to tc , and ge to ck ; therefore ck , ks , ge are all equal, as are also cg , gt , ke . Consequently the triangle $gte =$ the triangle kes , and each equal to half the constant inscribed parallelogram gk . And therefore the whole triangle cts , which is composed of the two smaller triangles and the parallelogram, is equal to double the constant inscribed parallelogram gk .

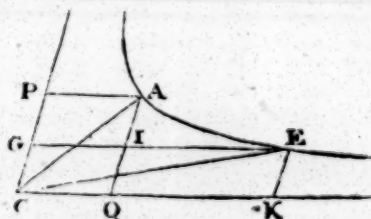
Q.E.D.

PRO-

PROPOSITION XXVIII.

The three following Spaces, between the Asymptotes and the Curve, are equal; namely, the Sector or Trilinear Space contained by an Arc of the Curve and two Radii, or Lines drawn from its Extremities to the Center, and each of the two Quadrilaterals, contained by the said Arc, and two Lines drawn from its Extremities parallel to one Asymptote, and the intercepted Part of the other Asymptote.

That is, the sector $CAE = PAEG = QAEK$,
all standing upon the same arc AE .



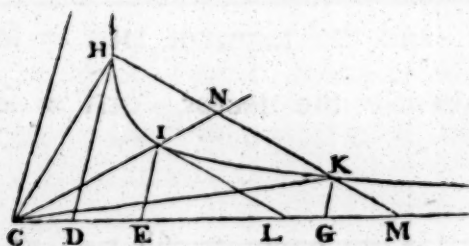
For, by prop. 26, $CPAQ = CGEK$;
subtract the common space $CGIQ$,
so shall the paral. $PI =$ the paral. IK ;
to each add the trilineal IAE ,
then is the quadr. $PAEG = QAEK$.
Again, from the quadrilateral $CAEK$
take the equal triangles CAQ , CEK ,
and there remains the sector $CAE = QAEK$.
Therefore $CAE = QAEK = PAEG$. Q.E.D.

P R O-

PROPOSITION XXIX.

If from the Point of Contact of any Tangent, and the two Intersections of the Curve with a Line parallel to the Tangent, three parallel Lines be drawn in any Direction, and terminated by either Asymptote; those three Lines shall be in continued Proportion.

That is, if HKM and the tangent IL be parallel,
then are the parallels DH, EI, GK in continued proportion.



For, by the parallels, $EI : IL :: DH : HM$;
and, by the same, $EI : IL :: GK : KM$;
theref. by compof. $EI^2 : IL^2 :: DH \cdot GK : HMK$;

but, by prop. 24, the rect. $HMK = IL^2$;

and theref. the rect. $DH \cdot GK = EI^2$,

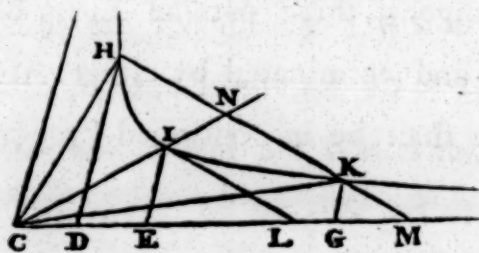
or

$DH : EI :: EI : GK.$ Q.E.D.

PRO-

PROPOSITION XXX.

Draw the semi-diameters CH , CIN , CK ;
Then shall the sector $CHI =$ the sector CIK .



For, because HK and all its parallels are bisected
by CIN , therefore the triangle $CNH =$ tri. CNK ,
and the segment $INH =$ seg. INK ;
consequently the sector $CIH =$ sec. CIK .

COROL. 2. If the geometricals DH , EI , GK be parallel
to the other asymptote, the spaces $DHIE$, $EIKG$ will be
equal; for they are equal to the equal sectors CHI ,
 CIK .

So that by taking any geometricals CD , CE , CG , &c,
and drawing DH , EI , GK , &c, parallel to the other asymp-
tote, as also the radii CH , CI , CK .

then

then the sectors CHI , CIK , &c,
or the spaces $DHIE$, $EIKG$, &c,
will be all equal among themselves.

Or the sectors CHI , CHK , &c,
or the spaces $DHIE$, $DHKG$, &c,
will be in arithmetical progression.

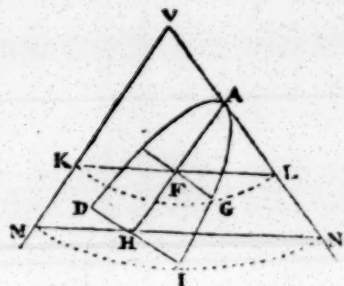
And therefore these sectors, or spaces, will be analogous
to the logarithms of the lines or bases CD , CE , CG , &c ;
namely CHI or $DHIE$ the log. of the ratio of
 CD to CE , or of CE to CG , &c ; or of EI to DH , or of GK to EI , &c ;
and CHK or $DHKG$ the log. of the ratio of
 CD to CG , &c, or of GK to DH , &c.

OF THE PARABOLA.

PROPOSITION I.

The Abscisses are Proportional to the Squares of their Ordinates.

Let AVM be a section through the axis of the cone, and $AGIH$ a parabolic section by a plane perpendicular to the former, and parallel to the side VN of the cone; also let AFH be the common intersection of the two planes, or the axis of the parabola, and FG , HI ordinates perpendicular to it.



Then I say that $AF : AH :: FG^2 : HI^2$.

For, through the ordinates FG , HI draw the circular sections KGL , MIN parallel to the base of the cone, having KL , MN for their diameters, to which FG , HI are ordinates, as well as to the axis of the parabola.

Then by similar triangles $AF : AH :: FL : HN$;

but, because of the parallels, $KF = MH$;

therefore $AF : AH :: KF \cdot FL : MH \cdot HN$.

But, by the circle, $KF \cdot FL = FG^2$, and $MH \cdot HN = HI^2$;

Therefore $AF : AH :: FG^2 : HI^2$. Q.E.D.

COROL.

COROL. 1. Hence the third proportional $\frac{EG^2}{AE}$ or $\frac{HI^2}{AH}$ is a constant quantity, and is equal to the parameter of the axis by cor. 5 prop. 1 of the ellipse.

Or $AE : EG :: EG : P$ the parameter.

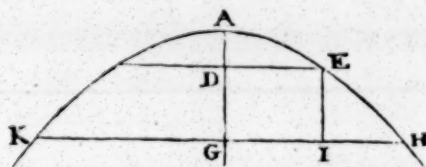
Or the rectangle $P \cdot AE = EG^2$

COROL. 2. Hence also as the property is the same for the ordinates on both sides of the diameter, it follows that the ordinates DH , HI , on both sides of the axis are equal, or that the double ordinates are bisected by the axis; and that the whole figure, made up of all the double ordinates, is also bisected by the axis.

PROPOSITION II.

As the Parameter of the Axis :
 Is to the Sum of any Two Ordinates ::
 So is the Difference of those Ordinates :
 To the Difference of their Abscisses :

That is, $P : GH + DE :: GH - DE : DG$,
 Or, $P : KI :: IH : EI$.



For, by cor. I prop. I, $P \cdot AG = GH^2$,
 and $P \cdot AD = DE^2$;
 theref. by subtrac. $P \cdot DG = GH^2 - DE^2$.

But $GH^2 - DE^2 = \overline{GH + DE} \cdot \overline{GH - DE} = KI \cdot IH$,
 theref. the rectangle $P \cdot DG = KI \cdot IH$,
 or $P : KI :: IH : DG$ or EI . Q.E.D.

COROL. Hence because $P \cdot EI = KI \cdot IH$,
 and, by cor. I prop. I, $P \cdot AG = GH^2$,
 therefore $AG : EI :: GH^2 : KI \cdot IH$.

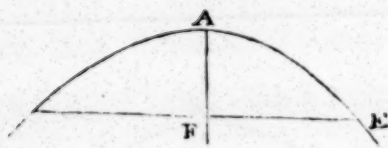
And so any diameter EI is as the rectangle of the segments KI, IH of the double ordinate KH.

PRO-

PROPOSITION III.

The Distance from the Vertex to the Focus is equal to $\frac{1}{4}$ of the Parameter, or to Half the Ordinate at the Focus.

That is, $AF = \frac{1}{2}FE = \frac{1}{4}P.$



For, the general property is $AF : FE :: FE : P.$

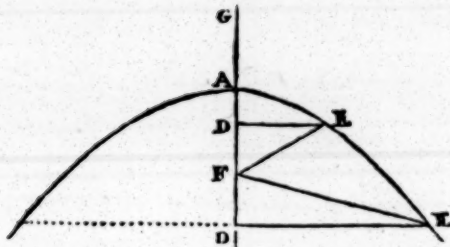
But, by the definition, $FE = \frac{1}{2}P;$

therefore also $AF = \frac{1}{2}FE = \frac{1}{4}P.$ Q.E.D.

PROPOSITION IV.

A Line drawn from the Focus to any Point in the Curve, is equal to the Sum of the Focal Distance and the Absciss of the Ordinate to that Point.

That is, $FE = FA + AD = GD$, taking $AG = AF$.



For since $FD = AD \cup AF$,

theref. by squaring, $FD^2 = AF^2 - 2AF \cdot AD + AD^2$,

But, by cor. I prop. I, $DE^2 = P \cdot AD = 4AF \cdot AD$;

theref. by addition, $FD^2 + DE^2 = AF^2 + 2AF \cdot AD + AD^2$,

But, by rt. angled Δ s, $FD^2 + DE^2 = FE^2$;

therefore $FE^2 = AF^2 + 2AF \cdot AD + AD^2$,

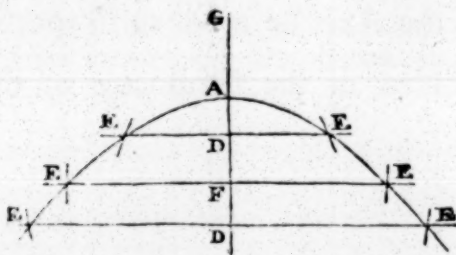
and the root or side is $FE = AF + AD$,

or $FE = GD$, by taking $AG = AF$. Q.E.D.

COROL.

COROL. 1. The difference $FE - FE$ is always equal to DD the difference of the abscisses.

COROL. 2. Hence also the curve is easily described

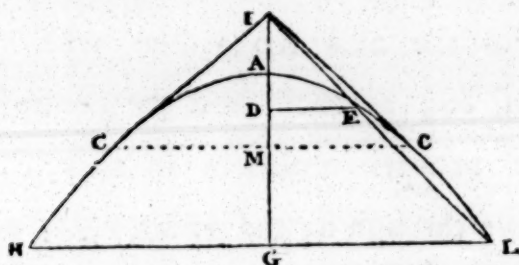


by points. Namely, in the axis produced take $AG = AF$ the focal distance, and draw a number of lines EE perpendicular to the axis AD ; then with the distances $GD, GD, GD, \&c.$ as radii, and the centre F , draw arcs crossing the parallel ordinates in $E, E, E, \&c.$ Then draw the curve through all the points $E, E, E.$

PROPOSITION V.

If a Line be drawn from any Point in the Axis produced, to cut the Curve in two Points; then shall the external Part of the Axis be a Mean Proportional between the two Abscisses of the Ordinates to the two Points of Interfection.

That is, AI a mean proportional between AD, AG,
or, $AD : AI :: AI : AG$.

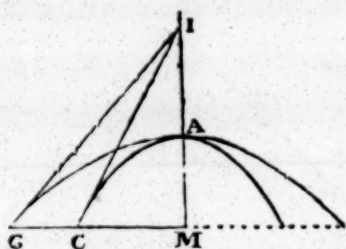


For, by prop. I, $AD : AG :: DE^2 : GH^2$;
and, by sim. tri. $ID^2 : IG^2 :: DE^2 : GH^2$ *;
theref. by equal. $AD : AG :: ID^2 : IG^2$;
and, by division, $AD : DG :: ID^2 : IG^2 - ID^2$ or $DG \cdot ID + IG$
or $AD : ID :: ID : ID + IG$;
and by division $AD : AI :: ID : IG$ *;
and again by div. $AD : AI :: AI : AG$. Q.E.D.

COROL. I. * Hence we have $AD : AI :: DE : GH$.

COROL.

COROL. 2. If the line be supposed to revolve about the point I ; then as it recedes farther from the axis, the points E and H approach towards each other, the point E descending, and the point H ascending, till at last they meet in the point C when the line becomes a tangent to the curve at C . And then the points D and G meet in the point M , and the ordinates DE , GH in the ordinate CM . Consequently AD , AG , becoming each equal to AM , their mean proportional AI will be equal to the absciss AM . That is, the external part of the axis, cut off by a tangent, is equal to the absciss of the ordinate to the point of contact.



COROL. 3. Hence the tangents to all parabolas, which have the same absciss, meet the axis produced in the same point. For if the absciss AM be the same in all, the external axis AI , which is equal to it, will be the same also.

COROL. 4. And hence also a tangent is easily drawn to the curve.

For if the point of contact C be given; draw the ordinate CM , and produce MA till $AI = AM$; then join IC the tangent.

Or if the point I be given; take $AM = AI$, and draw the ordinate MC , which will give the point of contact C , to which draw IC the tangent.

COROL. 2. The subnormal DG is every where equal to the constant quantity $2FA$, or FT , or $\frac{1}{2}P$ the semi-parameter.

For draw the tangent AH parallel to DC, making the triangle FHA similar to GCD.

Then $DC = 2AH$, because $DT = 2DA$:
consequently $DG = 2FA = \frac{1}{2}P$.

COROL. 3. The tangent at the vertex AH, is a mean proportional between AF and AD.

For because FHT is a right angle,
therefore AH is a mean between AF, AT,
or between AF, AD,
because $AD = AT$.
Likewise FH is a mean between FA, FT,
or between FA, FC.

COROL. 4. The tangent TC makes equal angles with FC and the axis FT, or with the line ICK drawn parallel to the axis.

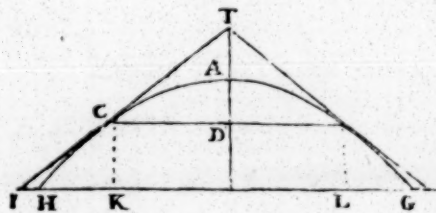
For because $FT = FC$,
therefore the $\angle FCT = \angle FTC =$ its altern. $\angle ICT$.
Also the angle $GCF =$ the angle GCK .

COROL. 5. And because the angle of incidence GCK is = the angle of reflection GCF ; therefore a ray of light falling upon the curve in the direction KC, will be reflected to the focus F. That is, all rays parallel to the axis, are reflected to the focus, or burning point.

PROPOSITION VII.

If an Ordinate be drawn to the Point of Contact of any Tangent, and another Ordinate produced to cut the Tangent; It will be as the Difference of the Ordinates : Is to the Difference added to the external Part :: So is Double the first Ordinate : To the Sum of the Ordinates.

That is, $KH : KI :: KL : KG$.



For, by cor. 1 prop. 1, $P : DC :: DC : DA$,
and $P : 2DC :: DC : DT$ or $2DA$.

But, by sim. triangles, $KI : KC :: DC : DT$;

therefore by equality, $P : 2DC :: KI : KC$,

or, $P : KI :: KL : KC$.

Again, by prop. 2, $P : KH :: KG : KC$;

therefore by equality, $KH : KI :: KL : KG$. Q.E.D.

COROL. 1. Hence, by composition and division,

we have $KH : KI :: GK : GI$,

and $HI : HK :: HK : KL$,

also $IH : IK :: IK : IG$; that is,

IK is a mean proportional between IG and IH .

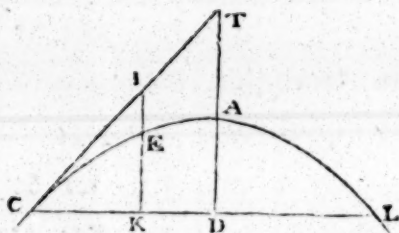
COROL. 2. And from this last property we can easily draw a tangent to the curve from any given point I . Namely, draw IHG perpendicular to the axis, and take IK a mean proportional between IH , IG ; then draw KC parallel to the axis, and c will be the point of contact, through which and the given point I the tangent IC is to be drawn.

PRO-

PROPOSITION VIII.

If there be any Tangent, and a Double Ordinate drawn from the Point of Contact, and also any Line parallel to the Axis, limited by the Tangent and Double Ordinate: then shall the Curve divide that Line in the same Ratio, as the Line divides the Double Ordinate.

That is, $IE : EK :: CK : KL$.

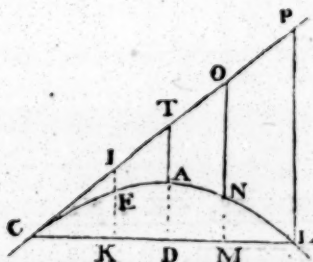


For, by sim. triangles, $CK : KI :: CD : DT$ or $2DA$;
 but, by the def. the param. $P : CL :: CD : 2DA$;
 therefore by equality, $P : CK :: CL : KI$.
 But, by prop. 2, $P : CK :: KL : KE$;
 therefore by equality, $CL : KL :: KI : KE$;
 and, by division, $CK : KL :: IE : EK$. Q.E.D.

PROPOSITION IX.

The fame being supposed as in Prop. VIII : then shall the External Part of the Line between the Curve and Tangent, be proportional to the Square of the intercepted Part of the Tangent, or to the Square of the intercepted Part of the Double Ordinate.

That is, IE is as CI^2 or as CK^2 .
 and IE, TA, ON, PL, &c,
 are as CI^2 , CT^2 , CO^2 , CP^2 , &c,
 or as CK^2 , CD^2 , CM^2 , CL^2 , &c.



For, by prop. 8, $IE : EK :: CK : KL$,
 or, by equality, $IE : EK :: CK^2 : CK \cdot KL$.
 But, by cor. prop. 2, EK is as the rect. $CK \cdot KL$,
 and therefore IE is as CK^2 , or as CI^2 . Q.E.D.

COROL.

COROL. 1. As this property is common to every position of the tangent, if the lines IE , TA , ON , &c, be appended to the points I , T , O , &c, and movable about them, and of such lengths as that their extremities E , A , N , &c, be in the curve of a parabola in some one position of the tangent; then making the tangent revolve about the point C , the extremities E , A , N , &c, will always form the curve of some parabola, in every position of the tangent.

COROL. 2. The parameter of the axis is also a third proportional to IE and CK .

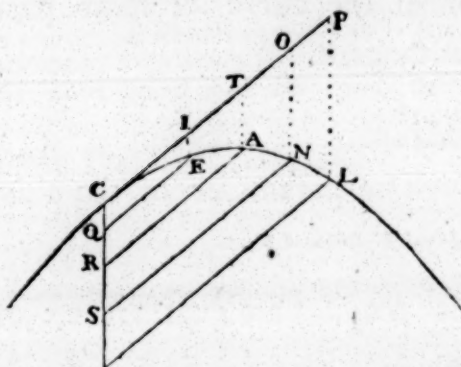
For, by this prop. $EK : KL :: IE : CK$;
 and, by prop. 2, $EK : KL :: CK : P$;
 therefore by equality, $IE : CK :: CK : P$.

PROPOSITION X.

The Abscisses of any Diameter, are as the Squares of their Ordinates.

That is, CQ , CR , CS , &c,
are as QE^2 , RA^2 , SN^2 , &c.

Or $CQ : CR :: QE^2 : RA^2$, &c.



For, draw the tangent CT , and the externals CI , AT , NO , &c, parallel to the axis, or to the diameter CS .

Then, because the ordinates QE , RA , SN , &c, are parallel to the tangent CT , by the definition of them, therefore all the figures IQ , TR , OS , &c are parallelograms, whose opposite sides are equal,

namely IE , TA , ON , &c,
are equal to CQ , CR , CS , &c.

Therefore by prop. 9, CQ , CR , CS , &c,
are as CI^2 , CT^2 , CO^2 , &c,

or as their equals QE^2 , RA^2 , SN^2 , &c. Q.E.D.

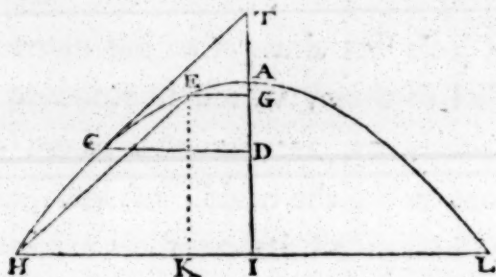
COROL. Here, like as in prop. 2, the difference of the abscisses, are as the difference of the squares of their ordinates, or as the rectangles under the sum and difference of the ordinates, the rectangle under the sum and difference of the ordinates being equal to the rectangle under the difference of the abscisses and the parameter of that diameter, or a third proportional to any absciss and its ordinate.

PRO-

PROPOSITION XI.

If a Line be drawn parallel to any Tangent, and cut the Curve in two Points; then if two Ordinates be drawn to the Intersections, and a third to the Point of Contact, these three Ordinates will be in Arithmetical Progression, or the Sum of the Extremes will be equal to Double the Mean.

That is $EG + HI = 2CD$.

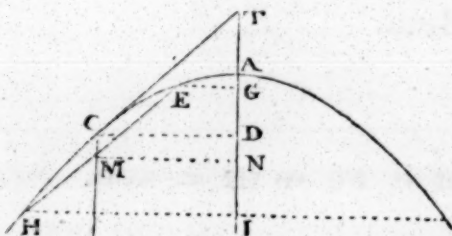


For draw EK parallel to the axis, and produce HI to L .
 Then, by sim. triangles, $EK : HK :: TD$ or $2AD : CD$;
 but, by prop. 2, $EK : HK :: KL : P$ the param.
 theref. by equality, $2AD : KL :: CD : P$.
 But, by the defin. $2AD : 2CD :: CD : P$;
 theref. the 2d terms are equal, $KL = 2CD$,
 that is $EG + HI = 2CD$. Q.E.D.

PROPOSITION XII.

Any Diameter bisects all its Double Ordinates, or Lines parallel to the Tangent at its Vertex.

That is, $ME = MH$.



For to the axis AI draw the ordinates EG, CD, HI, and MN parallel to them, which is equal to CD.

Then, by prop. XI, $2MN$ or $2CD = EG + HI$,
therefore M is the middle of EH.

And for the same reason all its parallels are bisected. Q.E.D.

SCHOLIUM.

Hence as the abscisses of any diameter and their ordinates have the same relations as those of the axis, namely, that the ordinates are bisected by the diameter, and their squares proportional to the abscisses; so all the other properties of the axis and its ordinates and abscisses, before demonstrated, will likewise hold good for any diameter and its ordinates and abscisses. And also those of the parameters, understanding the parameter of any diameter, as a third proportional to any absciss and its ordinate. Some of the most material of which are demonstrated in the following propositions.

The Parameter of any Diameter is equal to four Times the Line drawn from the Focus to the Vertex of that Diameter.

Then the absciffes AD, CM or AT being equal by cor. 2 to prop. 5, the parameters will be as the fquares of the ordinates CD, MA or CT, by the definition;

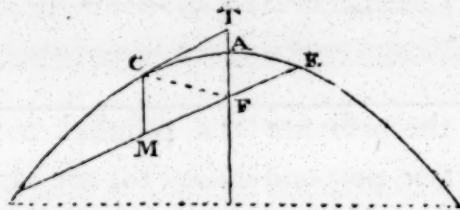
Q.E.D.

PRO-

PROPOSITION XIV.

If an Ordinate to any Diameter pass through the Focus, it will be equal to Half its Parameter; and its Absciss equal to One Fourth of the same Parameter.

That is, $CM = \frac{1}{4}p$, and $ME = \frac{1}{2}p$.



For join FC , and draw the tangent CT .

By the parallels, $CM = FT$;

and, by prop. 6, $FC = FT$;

also, by the last prop. $FC = \frac{1}{4}p$;

therefore $CM = \frac{1}{4}p$.

Again, by the defin. CM or $\frac{1}{4}p : ME :: ME : p$,

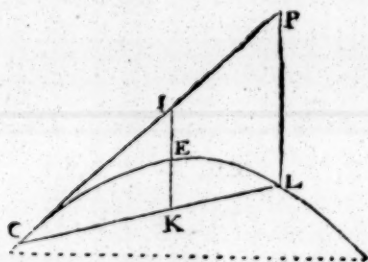
and consequently $ME = \frac{1}{2}p = 2CM$.

COROL. Hence, of any diameter, the double ordinate which passeth through the focus, is equal to the parameter, or to quadruple its absciss.

PROPOSITION XV.

If there be a Tangent, and any Line drawn from the Point of Contact and meeting the Curve in some other Point, as also another Line parallel to the Axis, and limited by the First Line and the Tangent: then shall the Curve divide this Second Line in the same Ratio, as the Second Line divides the First Line.

That is, $IE : EK :: CK : KL$.



For draw LP parallel to IK or to the axis.

Then, by prop. 9, $IE : PL :: CI^2 : CP^2$,

or by sim. tri. $IE : PL :: CK^2 : CL^2$.

Also, by sim. tri. $IK : PL :: CK : CL$,

or $IK : PL :: CK^2 : CK \cdot CL$;

therefore by equality, $IE : IK :: CK \cdot CL : CL^2$,

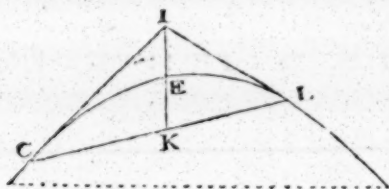
or $IE : IK :: CK : CL$;

and, by division, $IE : EK :: CK : KL$. Q.E.D.

PROPOSITION XVI.

If a Tangent cut any Diameter produced, and if an Ordinate to that Diameter be drawn from the Point of Contact; then the Distance in the Diameter produced, between the Vertex and the Intersection of the Tangent, will be equal to the Absciss of that Ordinate.

That is, $IE = EK$.



For, by the last prop. $IE : EK :: CK : KL$.

But, by prop. 12, $CK = KL$,

and therefore $IE = EK$.

Q.E.D.

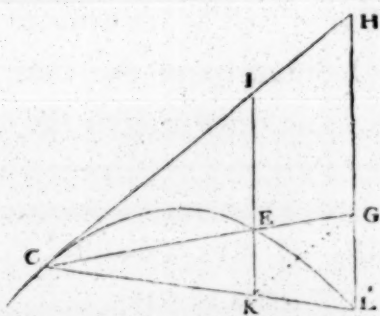
COROL. 1. The two tangents CI , LI , at the extremities of any double ordinate CL , meet in the same point of the diameter of that double ordinate produced. And the diameter drawn through the intersection of two tangents, bisects the line connecting the points of contact.

COROL. 2. Hence we have another method of drawing a tangent from any given point I without the curve. Namely, from I draw the diameter IK , in which take $EK = EI$, and through K draw CL parallel to the tangent at E ; then C and L are the points to which the tangents must be drawn from I .

PRO-

PROPOSITION XVII.

If from any Point of the Curve there be drawn a Tangent, and also Two Right Lines to cut the Curve; and Diameters be drawn through the Points of Intersection E and L , meeting those Two Right Lines in two other Points G and K : Then will the Line KG joining these last Two Points be parallel to the Tangent.



For, by prop. 15, $CK : KL :: EI : EK$;
 and by comp. $CK : CL :: EI : KI :: GH : LH$ by parallels,
 But, by sim. tri. $CK : CL :: KI : LH$;
 theref. by equal. $KI : LH :: GH : LH$;
 consequently $KI = GH$,
 and therefore KG is parallel and equal to IH . Q.E.D.

OTHERWISE.

By prop. 15, $EI : EK :: CK : KL :: CE : EG$ by the parallels. So that the two triangles CEI , KEG have their angles at E equal, and the sides about those angles proportional; they are therefore similar; consequently the angle at C is equal to the angle at G , and KG parallel to CH . Q.E.D.

COROL. 1. Hence we have $KI = GH$.

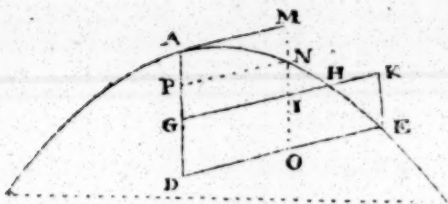
COROL. 2. Also KI or GH is a mean proportional between EI and LH .

For, by the parallels, $EI : KI :: GH$ or $KI : LH$.

PROPOSITION XVIII.

If a Line be drawn from the Vertex of any Diameter to cut the Curve in some other Point, and an Ordinate of that Diameter be drawn to that Point, as also another Ordinate any where cutting the Line, both produced if necessary: The Three will be continual Proportionals, namely, the two Ordinates and the Part of the Latter limited by the said Line drawn from the Vertex.

That is, DE, GH, GI are continual Proportionals,
Or $DE : GH :: GH : GI$.



For, by prop. 10,

$$DE^2 : GH^2 :: AD : AG ;$$

and, by sim. tri.

$$DE : GI :: AD : AG ;$$

theref. by equality

$$DE : GI :: DE^2 : GH^2,$$

that is, of the three DE, GH, GI, $1ft : 3d :: 1ft^2 : 2d^2$;

therefore

$$1ft : 2d :: 2d : 3d,$$

that is

$$DE : GH :: GH : GI. \text{ Q.E.D.}$$

COROL. I. Or their equals GK, GH, GI are proportionals; where EK is parallel to the diameter AD.

COROL.

COROL. 2. Hence we have $DE : AG :: p : GI$, where p is the parameter, or $AG : GI :: DE : p$.

For, by the defin. $AG : GH :: GH : p$.

COROL. 3. Hence also the three MN , MI , MO are proportionals, where MO is parallel to the diameter, and AM parallel to the ordinates.

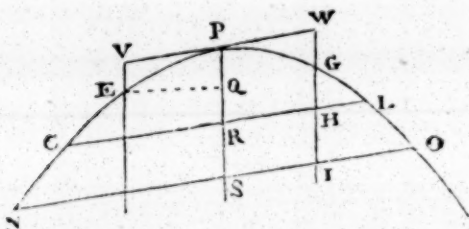
For, by prop. 10,	MN , MI , MO ,
or their equals	AP , AG , AD ,
are as the squares of	PN , GH , DE ,
or of their equals	GI , GH , GK ,
which are proportionals by cor. 1.	

PROPOSITION XIX.

If a Diameter cut any Parallel Lines terminated by the Curve; the Segments of the Diameter will be as the Rectangle of the Segments of those Lines.

That is, $EK : EM :: CK \cdot KL : NM \cdot MO$.

Or, EK is as the rectangle $CK \cdot KL$.



For draw the diameter PS to which the parallels CL , NO are ordinates, and the ordinate EQ parallel to them.

Then CK is the difference, and KL the sum of the ordinates EQ , CR ; also NM the difference, and MO the sum of the ordinates EQ , NS . And the differences of the abscissas, are QR , QS , or EK , EM .

Then by cor. prop. 10, $QR : QS :: CK \cdot KL : NM \cdot MO$,
that is $EK : EM :: CK \cdot KL : NM \cdot MO$.

COROL. I. The rect. $CK \cdot KL =$ rect. EK and the param. of PS .

For the rect. $CK \cdot KL =$ rect. QR and the param. of PS .

COROL.

COROL. 2. If any line CL be cut by two diameters EK , GH ; the rectangles of the parts of the line, are as the segments of the diameters.

For EK is as the rectangle $EK \cdot KL$,
and GH is as the rectangle $CH \cdot HL$;
therefore $EK : GH :: CK \cdot KL : CH \cdot HL$.

COROL. 3. If two parallels CL , NO be cut by two diameters EM , GI ; the rectangles of the parts of the parallels will be as the segments of the respective diameters.

For $EK : EM :: CK \cdot KL : NM \cdot MO$,
and $EK : GH :: CK \cdot KL : CH \cdot HL$,
theref. by equal. $EM : GH :: NM \cdot MO : CH \cdot HL$.

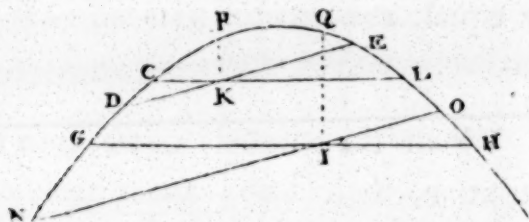
COROL. 4. When the parallels come into the position of the tangent at P , their two extremities, or points in the curve, unite in the point of contact P ; and the rectangle of the parts becomes the square of the tangent, and the same properties still follow them.

So that, $EV : PV :: PV : p$ the param.
 $GW : PW :: PW : p$,
 $EV : GW :: PV^2 : PW^2$,
 $EV : GH :: PV^2 : CH \cdot HL$.

PROPOSITION XX.

If two Parallels intersect any other two Parallels;
the Rectangles of the Segments will be respectively
Proportional.

That is, $CK \cdot KL : DK \cdot KE :: GI \cdot IH : NI \cdot IO$.



For, by cor. 3 prop. 19,

$$PK : QI :: CK \cdot KL : GI \cdot IH;$$

and by the same,

$$PK : QI :: DK \cdot KE : NI \cdot IO;$$

theref. by equal.

$$CK \cdot KL : DK \cdot KE :: GI \cdot IH : NI \cdot IO.$$

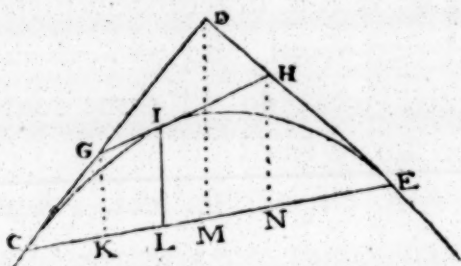
COROL. When one of the pairs of intersecting lines comes into the position of their parallel tangents meeting and limiting each other, the rectangles of their segments become the squares of their respective tangents. So that the constant ratio of the rectangles, is that of the square of their parallel tangents, namely,

$$CK \cdot KL : DK \cdot KE :: \text{sq. tang. parallel to CL} : \text{sq. tang. parallel to DE}.$$

PROPOSITION XXI.

If there be Three Tangents intersecting each other; their Segments will be in the same Proportion.

That is, $GI : IH :: CG : GD :: DH : HE$.



For through the points G, I, D, H , draw the diameters GK, IL, DM, HN ; as also the lines CI, EI , which are double ordinates to the diameters GK, HN , by cor. 1 prop. 16; therefore the diameters GK, DM, HN ,

bisect the lines CL, CE, LE ;

hence $KM = CM - CK = \frac{1}{2}CE - \frac{1}{2}CL = \frac{1}{2}LE = LN$ or NE ,

and $MN = ME - NE = \frac{1}{2}CE - \frac{1}{2}LE = \frac{1}{2}CL = CK$ or KL .

But, by parallels, $GI : IH :: KL : LN$,

and $CG : GD :: CK : KM$,

also $DH : HE :: MN : NE$.

But the 3d terms KL, CK, MN are all equal;

as also the 4th terms LN, KM, NE .

Therefore the first and second terms, in all the lines, are proportional, namely $GI : IH :: CG : GD :: DH : HE$. Q.E.D.

PRAC-

PRACTICAL EXERCISES

IN

MENSURATION.

QUEST. 1. **W**HAT difference is there between a floor 28 feet long by 20 broad, and two others each of half the dimensions; and what do all three come to at 45s per square, or 100 square feet?

Anf. dif. 280 sq. feet. Amount 18 guineas.

QU. 2. An elm plank is 14 feet 3 inches long, and I would have just a square yard flit off it; at what distance from the edge must the line be struck? Anf. $7\frac{22}{71}$ inches.

QU. 3. A ceiling contains 114 yards 6 feet of plastering, and the room 28 feet broad; what was the length of it? Anf. $36\frac{6}{7}$ feet.

QU. 4. A common joist is 7 inches deep, and $2\frac{1}{2}$ thick; but I want a scantling just as big again, that shall be 3 inches thick; what will the other dimension be?

Anf. $11\frac{2}{3}$ inches.

QU. 5. A wooden trough cost me 3s 6d painting within, at 6d per yard; the length of it was 102 inches, and the depth 21 inches; what was the width?

Anf. $27\frac{1}{4}$ inches.

Qu. 6. If my court yard be 47 feet 9 inches square, and I have laid a foot-path with Purbeck stone, of 4 feet wide, along one side of it; what will paving the rest with flints come to at 6d per square yard? Anf. £ 5 16 0 $\frac{1}{2}$.

Qu. 7. A ladder, 40 feet long, may be so planted, that it shall reach a window 33 feet from the ground on one side of the street; and, by only turning it over, without moving the foot out of its place, it will do the same by a window 21 feet high on the other side: what is the breadth of the street? Anf. 56 feet 7 $\frac{3}{4}$ inches.

Qu. 8. The paving of a triangular court, at 18d per foot, came to 100l; the longest of the three sides was 88 feet; required the sum of the other two equal sides. Anf. 106.85 feet.

Qu. 9. There are two columns in the ruins of Persepolis left standing upright; the one is 64 feet above the plain, and the other 50: in a straight line between these stands an ancient small statue, the head of which is 97 feet from the summit of the higher, and 86 feet from the top of the lower column, the base of which measures just 76 feet to the center of the figure's base. Required the distance between the tops of the two columns. Anf. 157 feet nearly.

Qu. 10. The perambulator, or surveying wheel, is so contrived, as to turn just twice in the length of a pole, or 16 $\frac{1}{2}$ feet; required the diameter. Anf. 2.626 feet.

Qu. 11. In turning a one-horse chaise within a ring of a certain diameter, it was observed that the outer wheel made two turns while the inner made but one: the wheels were both 4 feet high; and, supposing them fixed at the statutable distance of 5 feet asunder on the axletree,

axletree, what was the circumference of the track described by the outer wheel? Ans. 63 feet nearly.

Qu. 12. What is the side of that equilateral triangle whose area cost as much paving at 8d a foot, as the palli-fading the three sides did at a guinea a yard?

Ans. 72.746 feet.

Qu. 13. In the trapezium ABCD are given $AB = 13$, $BC = 31\frac{1}{3}$, $CD = 24$, and $DA = 18$, also B a right angle; required the area. Ans. 410.122.

Qu. 14. A roof, which is 24 feet 8 inches by 14 feet 6 inches, is to be covered with lead at 8lb per square foot: what will it come to at 18s per cwt? Ans. £ 22 19 10 $\frac{1}{4}$.

Qu. 15. Having a rectangular marble slab, 58 inches by 27, I would have a square foot cut off parallel to the shorter edge; I would then have the like quantity divided from the remainder parallel to the longer side; and this alternately repeated, till there shall not be the quantity of a foot left: what will be the dimensions of the remaining piece? Ans. 20.7 inches by 6.086.

Qu. 16. Given two sides of an obtuse-angled triangle, which are 20 and 40 poles; required the third side that the triangle may contain just an acre of land.

Ans. 58.876 or 23.099.

Qu. 17. The end wall of a house is 24 feet 6 inches in breadth, and 40 feet to the eaves; $\frac{1}{3}$ of which is 2 bricks thick, $\frac{1}{3}$ more is $1\frac{1}{2}$ brick thick, and the rest 1 brick thick. Now the triangular gable rises 38 courses of bricks, 4 of which usually make a foot in depth, and this is but $4\frac{1}{2}$ inches, or half a brick thick: what will this piece of work come to at 5l 10s per statute rod?

Ans. £ 20 11 7 $\frac{1}{2}$.

Qu. 18. How

Qu. 18. How many bricks will it take to build a wall 10 feet high, and 500 feet long, and a brick and half thick, reckoning the brick 10 inches long, and 4 courses to the foot in height?

Anf. 36000.

Qu. 19. How many bricks will build a square pyramid, of 100 feet on each side at the base, and also 100 feet perpendicular height: the dimensions of a brick being supposed 10 inches long, 5 inches broad, and 3 inches thick.

Anf. 3840000.

Qu. 20. If from a right-angled triangle, whose base is 12, and perpendicular 16 feet, a line be drawn parallel to the perpendicular cutting off a triangle whose area is 24 square feet; required the sides of this triangle.

Anf. 6, 8, and 10.

Qu. 21. The ellipse in Grosvenor-square measures 840 links across the longest way, and 612 the shortest, within the rails: now the walls being 14 inches thick, what ground do they inclose; and what do they stand upon?

Anf. $\begin{cases} \text{inclose 4ac or 6p.} \\ \text{stand on } 1760\frac{1}{2} \text{ sq feet.} \end{cases}$

Qu. 22. If a round pillar, 7 inches over, have 4 feet of stone in it; of what diameter is the column, of equal length, that contains 10 times as much?

Anf. 22.136 inches.

Qu. 23. A circular fish-pond is to be made in a garden, that shall take up just half an acre; what must be the length of the cord that strikes the circle?

Anf. $27\frac{3}{4}$ yards.

Qu. 24. When a roof is of a true pitch, the rafters are $\frac{3}{4}$ of the breadth of the building: now supposing the eaves-boards to project 10 inches on a side, what will the

K

new

new ripping a house cost, that measures 32 feet 9 inches long, by 22 feet 9 inches broad on the flat, at 15s per square?

Anf. £ 8 15 9½.

Qu. 25. A cable which is 3 feet long, and 9 inches in compass, weighs 22 lb; what will a fathom of that cable weigh, which measures a foot about?

Anf. 78½ lb.

Qu. 26. My plumber has put 28 lb per square foot into a cistern 74 inches and twice the thickness of the lead long, 26 inches broad, and 40 deep; he has also put three stays across it within, 16 inches deep, of the same strength, and reckons 22s per cwt, for work and materials. I, being a mason, have paved him a workshop, 22 feet 10 inches broad, with Purbeck stone, at 7d per foot; and upon the balance I find there is 3s 6d due to him; what was the length of the workshop?

Anf. 32 f 0¾ inch.

Qu. 27. The distance of the centers of two circles, whose diameters are each 50, being given equal to 30; what is the area of the space inclosed by their circumferences?

Anf. 559·119.

Qu. 28. If 20 feet of iron railing weigh half a ton when the bars are an inch and quarter square, what will 50 feet come to at 3½d per lb, the bars being but ⅞ of an inch square?

Anf. £ 20 0 2.

Qu. 29. The area of an equilateral triangle, whose base falls on the diameter, and its vertex in the middle of the arc of a semicircle, is equal to 100: what is the diameter of the semicircle?

Anf. 26·32148.

Qu. 30. It is required to find the thickness of the lead in a pipe of an inch and quarter bore, which weighs 14 lb per yard in length; the cubic foot of lead weighing 11325 ounces.

Anf. ·20737 inches.

Qu. 31. Sup-

Qu. 31. Suppose the expence of paving a semicircular plot, at 2s 4d per foot, come to 10l, what is the diameter of it?

Anf. 14.7737.

Qu. 32. What is the length of a chord which cuts off $\frac{1}{3}$ of the area from a circle whose diameter is 289?

Anf. 278.6716.

Qu. 33. My plumber has set me up a cistern, and, his shop-book being burnt, he has no means of bringing in the charge, and I do not choose to take it down to have it weighed; but by measure he finds it contains $64\frac{3}{10}$ square feet, and that it is precisely $\frac{1}{8}$ of an inch in thickness. Lead was then wrought at 21l per fother of $19\frac{1}{2}$ cwt. It is required from these items to make out the bill, allowing $6\frac{1}{2}$ oz for the weight of a cubic inch of lead.

Anf. £ 4 11 2

Qu. 34. What will the diameter of a globe be, when the solidity and superficial content are expressed by the same number?

Anf. 6.

Qu. 35. A sack, that would hold 3 bushels of corn, is $22\frac{1}{2}$ inches broad when empty; what will another sack contain which, being of the same length, has twice its breadth or circumference?

Anf. 12 bushels.

Qu. 36. A carpenter is to put an oaken curb to a round well, at 8d per foot square: the breadth of the curb is to be $7\frac{1}{4}$ inches, and the diameter within $13\frac{1}{2}$ feet: what will be the expence?

Anf. 5s $2\frac{1}{4}$ d.

Qu. 37. A gentleman has a garden 100 feet long, and 80 feet broad; and a gravel walk is to be made of an equal width half round it: what must the breadth of the walk be to take up just half the ground?

K 2

Anf. 25.968 feet.

Q. 38. A

Qu. 38. A may-pole whose top, being broken off by a blast of wind, struck the ground at 15 feet distance from the foot of the pole; what was the height of the whole may-pole, supposing the length of the broken piece to be 39 feet?

Anf. 75 feet.

Qu. 39. Seven men bought a grinding stone of 60 inches diameter, each paying $\frac{1}{7}$ part of the expence; what part of the diameter must each grind down for his share?

Anf. the 1st 4.4508, 2d 4.8400, 3d 5.3535, 4th 6.0765, 5th 7.2079, 6th 9.3935, 7th 22.6778.

Qu. 40. A maltster has a kiln that is 16 feet 6 inches square: but he wants to pull it down, and build a new one that may dry three times as much at once as the old one; what must be the length of its side?

Anf. 28 f 7 inches.

Qu. 41. How many 3 inch cubes may be cut out of a 12 inch cube?

Anf. 64.

Qu. 42. How long must the tether of a horse be, that will allow him to graze, quite around, just an acre of ground?

Anf. $39\frac{1}{4}$ yards.

Qu. 43. What will the painting of a conical spire come to at 8d per yard; supposing the height to be 118 feet, and the circumference of the base 64 feet?

Anf. £ 14 0 $8\frac{3}{4}$.

Qu. 44. The diameter of a standard corn bushel is $18\frac{1}{2}$ inches, and its depth 8 inches; then what must the diameter of that bushel be whose depth is $7\frac{1}{2}$ inches?

Anf. 19.1067.

Qu. 45. Sup-

Qu. 45. Suppose the ball on the top of St. Paul's church is 6 feet in diameter; what did the gilding of it cost at $3\frac{1}{2}$ d per square inch?

Ans. £ 237 19 10 $\frac{1}{4}$.

Qu. 46. What will a frustum of a marble cone come to at 12s per solid foot; the diameter of the greater end being 4 feet, that of the less end $1\frac{1}{2}$, and the length of the slant side 8 feet?

Ans. £ 30 1 10 $\frac{1}{4}$.

Qu. 47. To divide a cone into three equal parts by sections parallel to the base, and to find the altitudes of the three parts, the height of the whole cone being 20 inches.

Ans. the upper part 13.867
the middle part 3.604
the lower part 2.528

Qu. 48. A gentleman has a bowling-green, 300 feet long, and 200 feet broad, which he would raise 1 foot higher, by means of the earth to be dug out of a ditch that goes round it: to what depth must the ditch be dug, supposing its breadth to be every where 8 feet?

Ans. $7\frac{2}{3}$ feet.

Qu. 49. How high above the earth must a person be raised, that he may see $\frac{1}{3}$ of its surface?

Ans. to the height of the earth's diameter.

Qu. 50. A cubic foot of brass is to be drawn into wire of $\frac{1}{40}$ of an inch in diameter; what will the length of the wire be, allowing no loss in the metal?

Ans. 97784.797 yards, or 55 miles 984.797 yards.

Qu. 51. Of what diameter must the bore of a cannon be, which is cast for a ball of 24 lb weight, so that the diameter of the bore may be $\frac{1}{10}$ of an inch more than that of the ball?

Ans. 5.757 inches.

Qu. 52. Supposing the diameter of an iron 9 lb ball to be 4 inches, as it is very nearly ; it is required to find the diameters of the several balls weighing 1, 2, 3, 4, 6, 12, 18, 24, 36, and 42 lb, and the caliber of their guns, allowing $\frac{1}{30}$ of the caliber, or $\frac{1}{49}$ of the ball's diameter, for windage.

Answer.

Wt ball	Diameter ball	Caliber gun
1	1.9230	1.9622
2	2.4228	2.4723
3	2.7734	2.8301
4	3.0526	3.1149
6	3.4943	3.5656
9	4.0000	4.0816
12	4.4026	4.4924
18	5.0397	5.1425
24	5.5469	5.6601
36	6.3496	6.4792
42	6.6844	6.8208

Qu. 53. Supposing the windage of all mortars be allowed to be $\frac{1}{80}$ of the caliber, and the diameter of the hollow part of the shell to be $\frac{7}{10}$ of the caliber of the mortar: It is required to determine the diameter and weight of the shell, and the quantity or weight of powder requisite to fill it, for each of the several sorts of mortars, namely, the 13, 10, 8, 5.8, and 4.6 inch mortar.

Answer.

Calib. mort.	Diameter shell	Wt shell empty	Wt of powder	Wt shell filled
4.6	4.523	8.320	0.583	8.903
5.8	5.703	16.677	1.168	17.845
8	7.867	43.764	3.065	46.829
12	9.833	85.476	5.986	91.462
13	12.783	187.791	13.151	200.942

Qu. 54. If a heavy sphere, whose diameter is 4 inches, be let fall into a conical glass, full of water, whose diameter is 5, and altitude 6 inches; it is required to determine how much water will run over.

Anf. 26.272 cubic inches, or near $\frac{35}{47}$ parts of a pint.

Qu. 55. The dimensions of the sphere and cone being the same as in the last question, and the cone only $\frac{1}{5}$ full of water; required what part of the axis of the sphere is immersed in the water: Anf. .546 parts of an inch.

Qu. 56. The cone being still the same, and $\frac{1}{5}$ full of water; required the diameter of a sphere which shall be just all covered by the water. Anf. 2.445996.

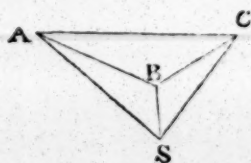
Qu. 57. If a person, with an air balloon, ascend vertically from London, to such height that he can just see Oxford appear in the horizon; it is required to determine his height above the earth, supposing the earth's radius to be 3965 miles, and the distance between London and Oxford 49.5933 miles.

Anf. $\frac{311}{1000}$ of a mile, or 547 yards 1 foot.

Qu. 58. In a garrison there are three remarkable objects A, B, C, the distances of which from one to another are known to be AB 213, AC 424, and BC 262 yards; I am desirous of knowing my position and distance at a place or station s, from whence I observed the angle ASB $13^{\circ} 30'$, and the angle CSB $29^{\circ} 50'$, both by geometry and trigonometry.

Answer.

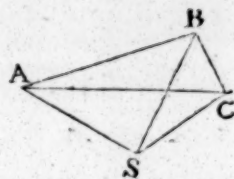
AS $605\frac{1}{2}$, BS $429\frac{1}{2}$, CS 524.



Qu. 59. Required the same as in the last question, when the point B is on the other side of AC, supposing AB 9, AC 12, and BC 6 furlongs; also the angle ASB $33^{\circ} 45'$, and the angle BSC $22^{\circ} 30'$.

Answer.

AS 10.64, BS 15.64, CS 14.01.



Qu. 60. It is required to determine the magnitude of a cube of gold, of the standard fineness, which shall be equal to the national debt of 240 million of pounds; supposing a guinea to weigh 5 dwts $9\frac{1}{2}$ grains.

Anf. 15.3006 feet.

Qu. 61. The ditch of a fortification is 1000 feet long, 9 feet deep, 20 feet broad at bottom, and 22 at top; how much water will fill the ditch?

Anf. 1158182 gallons nearly.

S P E C I F I C G R A V I T Y.

THE specific gravities of bodies, are their relative weights contained under the same given magnitude ; as a cubic foot, or a cubic inch, &c.

The specific gravities of several sorts of matter are expressed by the numbers annexed to their names in the following table.

A Table of the Specific Gravities of Bodies.

Fine gold	19640	Brick	2000
Standard gold	18888	Light earth	1984
Quick-silver	14000	Solid gun powder	1745
Lead	11325	Sand	1520
Fine silver	11091	Pitch	1150
Standard silver	10535	Box-wood	1030
Copper	9000	Sea-water	1030
Gun metal	8784	Common water	1000
Cast brass	8000	Oak	925
Steel	7850	Gun-powder, shaken	922
Iron	7645	Ash	800
Cast iron	7425	Maple	755
Tin	7320	Elm	600
Marble	2700	Fir	550
Common stone	2520	Cork	240
Loom	2160	Air at a mean state	1 $\frac{1}{2}$

Note.

Note. The several sorts of wood are supposed to be dry. Also, as a cubic foot of water weighs just 1000 ounces avoirdupois, the numbers in this table express, not only the specific gravities of the several bodies, but also the weight of a cubic foot of each, in avoirdupois ounces; and thence, by proportion, the weight of any other quantity, or the quantity of any other weight, may be known, as in the following problems.

PROBLEM I.

To find the Magnitude of any Body from its Weight.

As the tabular specific gravity of the body,
Is to its weight in avoirdupois ounces,
So is one cubic foot, or 1728 cubic inches,
To its content in feet, or inches, respectively.

EXAMPLES.

Ex. 1. Required the content of an irregular block of common stone which weighs 1 cwt, or 112 lb.

Anf. $1228\frac{2016}{2520}$ cub. inc.

Ex. 2. How many cubic inches of gun-powder are there in 1 lb weight.

Anf. 30 cubic inches nearly.

Ex. 3. How many cubic feet are there in a ton weight of dry oak?

Anf. $38\frac{138}{185}$ cubic feet,

PROBLEM II.

To find the Weight of a Body from its Magnitude.

As one cubic foot, or 1728 cubic inches,
Is to the content of the body,
So is its tabular specific gravity,
To the weight of the body.

EXAMPLES.

EXAMPLES.

Ex. 1. Required the weight of a block of marble, whose length is 63 feet, and breadth and thickness each 12 feet; being the dimensions of one of the stones in the walls of Balbeck.

Anf. $683\frac{4}{10}$ ton, which is nearly equal to the burthen of an East-India ship.

Ex. 2. What is the weight of 1 pint, ale measure, of gun-powder? Anf. 19 oz. nearly.

Ex. 3. What is the weight of a block of dry oak, which measures 10 feet in length, 3 feet broad, and $2\frac{1}{2}$ feet deep? Anf. $4335\frac{1}{8}$ lb.

PROBLEM III.

To find the Specific Gravity of a Body.

CASE 1. When the body is heavier than water, weigh it both in water and out of water, and take the difference, which will be the weight lost in water. Then

As the weight lost in water,
Is to the whole weight,
So is the specific gravity of water,
To the specific gravity of the body.

EXAMPLE.

A piece of stone weighed 10 lb, but in water only $6\frac{3}{4}$ lb, required its specific gravity. Anf. 3077.

CASE 2. When the body is lighter than water, so that it will not quite sink; affix to it a piece of another body heavier than water, so that the mass compounded of the two may sink together. Weigh the heavier body, and the compound mass, separately, both in water and out of it; then find how much each loses in water, by subtracting its weight

weight in water from its weight in air; and subtract the less of these remainders from the greater. Then

As this last remainder,
Is to the weight of the light body in air,
So is the specific gravity of water,
To the specific gravity of the body.

EXAMPLE.

Suppose a piece of elm weighs 15 lb in air, and that a piece of copper, which weighs 18 lb in air and 16 lb in water, is affixed to it, and that the compound weighs 8 lb in water; required the specific gravity of the elm.

Anf. 600.

PROBLEM IV.

To find the Quantities of Two Ingredients in a given Compound.

Take the three differences of every pair of the three specific gravities, namely, the specific gravities of the compound and each ingredient; and multiply the difference of every two specific gravities by the third. Then, as the greatest product is to the whole weight of the compound, so is each of the other products to the two weights of the ingredients,

EXAMPLE.

A composition of 112 lb being made of tin and copper, whose specific gravity is found to be 8784; required the quantity of each ingredient, the specific gravity of tin being 7320, and of copper 9000.

Answer, there is 100 lb of copper }
and conseq. 12 lb of tin } in the composition.

OF THE
WEIGHT AND DIMENSIONS
OF
BALLS AND SHELLS.

THE weight and dimensions of balls and shells might be found from the problems last given concerning specific gravity. But they may be found still easier by means of the experimented weight of a ball of a given size, from the known proportion of similar figures, namely, as the cubes of their diameters.

PROBLEM I.

To find the Weight of an Iron Ball, from its Diameter.

An iron ball of 4 inches diameter weighs 9 lb, and the weights being as the cubes of the diameters, it will be as 64 (which is the cube of 4) is to 9, so is the cube of the diameter of any other ball, to its weight. Or take $\frac{2}{64}$ of the cube of the diameter, for the weight. Or take $\frac{1}{8}$ of the cube of the diameter, and $\frac{1}{8}$ of that again, and add the two together, for the weight.

EXAMPLES.

Ex. 1. The diameter of an iron shot being 6.7, required its weight.

Ans. 42.294 lb.

Ex. 2.

Ex. 2. What is the weight of an iron ball whose diameter is 5.54 inches? Anf. 24 lb.

PROBLEM II.

To find the Weight of a Leaden Ball.

A leaden ball of $4\frac{1}{4}$ inches diameter weighs 17 lb; therefore as the cube of $4\frac{1}{4}$ to 17, or nearly as 9 to 2, so is the cube of the diameter of a leaden ball, to its weight.

Or take $\frac{2}{9}$ of the cube of the diameter, for the weight, nearly.

EXAMPLES.

Ex. 1. Required the weight of a leaden ball of 6.6 inches diameter. Anf. 63.888 lb.

Ex. 2. What is the weight of a leaden ball of 5.24 inches diameter? Anf. 32 lb nearly.

PROBLEM III.

To find the Diameter of an Iron Ball.

Multiply the weight by $7\frac{1}{9}$, and the cube root of the product will be the diameter.

EXAMPLES.

Ex. 1. Required the diameter of a 42 lb iron ball. Anf. 6.685 inches.

Ex. 2. What is the diameter of a 24 lb iron ball? Anf. 5.54 inches.

PROBLEM IV.

To find the Diameter of a Leaden Ball.

Multiply the weight by 9, and divide the product by 2; then the cube root of the quotient will be the diameter.

EXAMPLES.

EXAMPLES.

Ex. 1. Required the diameter of a 64 lb leaden ball.

Anf. 6.605 inches.

Ex. 2. What is the diameter of an 8 lb leaden ball?

Anf. 3.303 inches.

PROBLEM V.

To find the Weight of an Iron Shell.

Take $\frac{9}{64}$ of the difference of the cubes of the external and internal diameter, for the weight of the shell.

That is, from the cube of the external diameter take the cube of the internal diameter, multiply the remainder by 9, and divide the product by 64.

EXAMPLES.

Ex. 1. The outside diameter of an iron shell being 12.8, and the inside diameter 9.1 inches; required its weight.

Anf. 188.941 lb.

Ex. 2. What is the weight of an iron shell, whose external and internal diameters are 9.8 and 7 inches?

Anf. $84\frac{1}{4}$ lb.

PROBLEM VI.

To find how much Powder will fill a Shell. 

Divide the cube of the internal diameter, in inches, by 57.3, for the lbs of powder.

EXAMPLES.

Ex. 1. How much powder will fill the shell whose internal diameter is 9.1 inches? Anf. $13\frac{2}{3}$ lb nearly.

Ex. 2. How much powder will fill the shell whose internal diameter is 7 inches? Anf. 6 lb.

PROBLEM VII.

To find how much Powder will fill a Rectangular Box.

Find the content of the box in inches, by multiplying the length, breadth, and depth all together. Then divide by 30 for the pounds of powder.

EXAMPLES.

Ex. 1. Required the quantity of powder that will fill a box, the length being 15 inches, the breadth 12, and the depth 10 inches. Ans. 60 lb.

Ex. 2. How much powder will fill a cubical box whose side is 12 inches? Ans. $57\frac{3}{5}$ lb.

PROBLEM VIII.

To find how much Powder will fill a Cylinder.

Multiply the square of the diameter by the length, then divide by 38.2 for the pounds of powder.

EXAMPLES.

Ex. 1. How much powder will the cylinder hold whose diameter is 10 inches, and length 20 inches? Ans. $52\frac{1}{3}$ lb nearly.

Ex. 2. How much powder can be contained in the cylinder, whose diameter is 4 inches, and length 12 inches? Ans. $5\frac{5}{9}$ lb.

PROBLEM IX.

To find the Size of a Shell to contain a given Weight of Powder.

Multiply the pounds of powder by 57.3, and the cube root of the product will be the diameter in inches.

EXAMPLES.

Ex. 1. What is the diameter of a shell that will hold $13\frac{1}{8}$ lb of powder? Ans. 9.1 inches.

Ex.

Ex. 2. What is the diameter of a shell to contain 6 lb of powder? Ans. 7 inches.

PROBLEM X.

To find the Size of a Cubical Box to contain a given Weight of Powder.

Multiply the weight in pounds by 30, and the cube root of the product will be the side of the box in inches.

EXAMPLES.

Ex. 1. Required the side of a cubical box to hold 50 lb of gun-powder. Ans. 11.44 inches.

Ex. 2. Required the side of a cubical box to hold 400 lb of gun-powder. Ans. 22.89 inches.

PROBLEM XI.

To find what Length of a Cylinder will be filled by a given Weight of Gun-powder.

Multiply the weight in pounds by 38.2, and divide the product by the square of the diameter in inches, for the length.

EXAMPLES.

Ex. 1. What length of a 36 pounder gun, of $6\frac{2}{3}$ inches diameter, will be filled with 12 lb of powder? Ans. 10.314 inches.

Ex. 2. What length of a cylinder of 8 inches diameter may be filled with 20 lb of powder? Ans. $11\frac{1}{8}$ inc.

OF THE
P I L I N G
OF
B A L L S A N D S H E L L S.

IRON balls and shells are commonly piled, by horizontal courses, either in a pyramidal or wedge-like form; the base being either an equilateral triangle, a square, or a rectangle. In the triangle and square, the pile will finish in a single ball; but in the rectangle, it will finish in a single row of balls, like an edge.

In triangular and square piles, the number of horizontal rows, or courses, is always equal to the number of balls in one side of the bottom row. And in rectangular piles, the number of rows is equal to the number of balls in the breadth of the bottom row. Also in the number in the top row, or edge, is one more than the difference between the length and breadth of the bottom row.

P R O B L E M I.

To find the Number of Balls in a Triangular Pile.

Multiply continually together the number in one side of the bottom row, that number increased by 1, and the same number increased by 2; and $\frac{1}{6}$ of the last product will be the answer.

That

That is, $\frac{n \cdot n + 1 \cdot n + 2}{6}$ is the number or sum ; where

n is the number in the bottom row.

Ex. 1. Required the number of balls in a triangular pile, each side of the base containing 30 balls. Ans. 4960.

Ex. 2. How many balls are in the triangular pile, each side of the base containing 20? Ans. 1540.

PROBLEM II.

To find the Number of Balls in a Square Pile.

Multiply continually together the number in one side of the bottom course, that number increased by 1, and double the same number increased by 1 ; then $\frac{1}{6}$ of the last product will be the answer.

That is, $\frac{n \cdot n + 1 \cdot 2n + 1}{6}$ is the number.

EXAMPLES.

Ex. 1. How many balls are in a square pile of 30 rows? Ans. 9455.

Ex. 2. How many balls are in a square pile of 20 rows? Ans. 2870.

PROBLEM III.

To find the Number of Balls in a Rectangular Pile.

From 3 times the number in the length of the base row, subtract one less than the breadth of the same, multiply the remainder by the said breadth, and the product by one more than the same ; and divide by 6 for the answer.

That is, $\frac{b \cdot b + 1 \cdot 3l - b + 1}{6}$ is the number ; where

l is the length, and b the breadth of the lowest course.

Note. In all the piles, the breadth of the bottom is equal to the number of courses. And in the oblong or rectangular pile, the top row is one more than the difference between the length and breadth of the bottom.

EXAMPLES.

Ex. 1. Required the number of balls in a rectangular pile, the length and breadth of the base row being 46 and 15.
Anf. 4960.

Ex. 2. How many shot are in a rectangular complete pile, the length of the bottom course being 59, and its breadth 20?
Anf. 11060.

PROBLEM IV.

To find the Number of Balls in an Incomplete Pile.

From the number in the whole pile, considered as complete, subtract the number in the upper pile which is wanting at the top, both computed by the rule for their proper form; and the remainder will be the number in the frustum, or incomplete pile.

EXAMPLES.

Ex. 1. To find the number of shot in the incomplete triangular pile, one side of the bottom course being 40, and the top course 20.
Anf. 10150.

Ex. 2. How many shot are in the incomplete triangular pile, the side of the base being 24, and of the top 8?
Anf. 2516.

Ex. 3. How many balls are in the incomplete square pile, the side of the base being 24, and of the top 8?
Anf. 4760.

Ex. 4. How many shot are in the incomplete rectangular pile, of 12 courses, the length and breadth of the base being 40 and 20?
Anf. 6146.

OF
D I S T A N C E S
BY THE
V E L O C I T Y O F S O U N D.

BY various experiments it has been found that sound flies, through the air, uniformly at the rate of about 1142 feet in 1 second of time, or a mile in $4\frac{2}{3}$ seconds. And therefore, by proportion, any distance may be found corresponding to any given time; namely, multiply the given time, in seconds, by 1142, for the corresponding distance in feet; or take $\frac{3}{14}$ of the given time for the distance in miles.

Note. The time for the passage of sound in the interval between seeing the flash of a gun, or lightning, and hearing the report, may be observed by a watch, or a small pendulum. Or it may be observed by the beats of the pulse in the wrist, counting, on an average, about 70 to a minute for persons in moderate health, or $5\frac{1}{2}$ pulsations to a mile; and more or less according to circumstances.

E X A M P L E S.

Ex. 1. After observing a flash of lightning, it was 12 seconds before I heard the thunder; required the distance of the cloud from whence it came.

Ans. $2\frac{4}{7}$ miles.

L 3

Ex. 2.

Ex. 2. How long, after firing the tower guns, may the report be heard at Shooters-Hill, supposing the distance to be 8 miles in a straight line? Anf. $37\frac{1}{3}$ seconds.

Ex. 3. After observing the firing of a large cannon at a distance, it was 7 seconds before I heard the report; what was its distance? Anf. $1\frac{1}{2}$ mile.

Ex. 4. Perceiving a man at a distance hewing down a tree with an axe, I remarked that 6 of my pulsations passed between seeing him strike and hearing the report of the blow; what was the distance between us, allowing 70 pulses to a minute? Anf. 1 mile and 198 yards.

Ex. 5. How far off was the cloud from which thunder issued, whose report was 5 pulsations after the flash of lightning; counting 75 to a minute? Anf. 1523 yards.

Ex. 6. If I see the flash of a cannon fired by a ship in distress at sea, and hear the report 33 seconds after, how far is she off? Anf. $7\frac{1}{4}$ miles.

PRACTICAL EXERCISES

IN

MECHANICS, STATICS, HYDROSTATICS, SOUND, MOTION, GRAVITY, PROJECTILES, AND OTHER BRANCHES OF NATURAL PHILOSOPHY.

QUESTION 1. **R**EQUIRED the weight of a cast iron ball of 3 inches diameter, supposing the weight of a cubic inch of the metal to be 0.258 lb avoirdupois.

Anf. 3.64739 lb.

Qu. 2. To determine the weight of a hollow spherical iron shell 5 inches in diameter, the thickness of the metal being one inch.

Anf. 13.2387 lb.

Qu. 3. Being one day ordered to observe how far a battery of cannon was from me, I counted by my watch 17 seconds between the time of seeing the flash and hearing the report; what was the distance?

Anf. $3\frac{1}{2}$ miles.

Qu. 4. If the diameter of the earth be 7930 miles, and that of the moon 2160 miles; required the proportion of their surfaces, and also of their solidities; supposing them both to be globular, as they are very nearly.

Anf. the surfaces are as $13\frac{1}{2}$ to 1 nearly.
and the solidities as $49\frac{1}{2}$ to 1 nearly.

L 4

Qu. 5. It

Qu. 5. It is proposed to determine the proportional quantities of matter in the earth and moon; the density of the former being to that of the latter as 10 to 7, and their diameters as specified in the preceding problem.

Anf. as 71 to 1 nearly.

Qu. 6. What difference is there, in point of weight, between a block of marble containing 1 cubic foot and a half, and another of brass of the same dimensions?

Anf. 496 lb 14 oz.

Qu. 7. In the walls of Balbeck in Turkey, the ancient Heliopolis, there are three stones laid end to end, now in sight, that measure in length 61 yards; one of which in particular is 21 yards or 63 feet long, 12 feet thick, and 12 feet broad: now if this block be marble, what power would balance it, so as to prepare it for moving?

Anf. $693\frac{9}{10}$ tons, the burthen of an East-India ship.

Qu. 8. The battering-ram of Vespasian weighed, suppose 100000 pounds; and was moved, let us admit, with such a velocity, by strength of hands, as to pass through 20 feet in one second of time; and this was found sufficient to demolish the walls of Jerusalem. The question is with what velocity a 32 lb ball must move, to do the same execution.

Anf. 62500 feet.

Qu. 9. There are two bodies, of which the one contains 25 times the matter of the other, or is 25 times heavier; but the less moves with 1000 times the velocity of the greater: in what proportion then are the momenta or forces with which they are moved?

Anf. the less moves with a force 40 times greater.

Qu. 10. A body, weighing 20 lb, is impelled by such a force as to send it through 100 feet in a second; with what

what velocity then would a body of 8 lb weight move, if it were impelled by the same force?

Anf. 250 feet per second.

Qu. 11. There are two bodies, the one of which weighs 100 lb, the other 60; but the less body is impelled by a force 8 times greater than the other; the proportion of the velocities, with which these bodies move, is required.

Anf. the velocity of the greater to that of the less, as 3 to 40.

Qu. 12. There are two bodies, the greater contains 8 times the quantity of matter in the less, and is moved with a force 48 times greater; the ratio of the velocities of these two bodies is required.

Anf. the greater to the less, as 6 to 1.

Qu. 13. There are two bodies, one of which moves 40 times swifter than the other; but the swifter body has moved only one minute, whereas the other has been in motion 2 hours: the ratio of the spaces described by these two bodies is required.

Anf. the swifter to the slower, as 1 to 3.

Qu. 14. Supposing one body to move 30 times swifter than another, as also the swifter to move 12 minutes, the other only 1: what difference will there be between the spaces described by them, supposing the last has moved 60 inches?

Anf. 1795 feet.

Qu. 15. There are two bodies, the one of which has passed over 50 miles, the other only 5; and the first had moved with 5 times the celerity of the second: what is the ratio of the times they have been in describing those spaces?

Anf. as 2 to 1.

Qu. 16. If

Qu. 16. If a lever, 40 effective inches long, will, by a certain power thrown successively upon it, in 13 hours, raise a weight 104 feet; in what time will two other levers, each 18 effective inches long, raise an equal weight 73 feet?

Anf. 10 hours $8\frac{1}{3}$ minutes.

Qu. 17. What weight will a man be able to raise, who presses with the force of a hundred and a half, on the end of an equipoised handspike 100 inches long, meeting with a convenient prop exactly $7\frac{1}{2}$ inches from the lower end of the machine?

Anf. 2072 lb.

Qu. 18. A weight of $1\frac{1}{2}$ lb laid on the shoulder of a man is no greater burthen to him than its absolute weight, or 24 ounces: what difference will he feel, between the said weight applied near his elbow, at 12 inches from the shoulder, and in the palm of his hand, 28 inches from the same; and how much more must his muscles then draw to support it at right angles, that is, having his arm stretched right out?

Anf. 24 lb avoirdupois.

Qu. 19. What weight hung on at 70 inches from the centre of motion of a steel-yard, will balance a small gun of $9\frac{1}{2}$ cwt, freely suspended at 2 inches distance from the said centre on the contrary side?

Anf. $30\frac{2}{3}$ lb.

Qu. 20. It is proposed to divide the beam of a steel-yard, or to find the points of division where the weights of 1, 2, 3, 4, &c lb on the one side, will just balance a constant weight of 95 lb at the distance of 2 inches on the other side of the fulcrum, the weight of the beam being 10 lb, and its whole length 36 inches.

Anf. 30, 15, 10, $7\frac{1}{3}$, 6, 5, $4\frac{2}{7}$, $3\frac{3}{4}$, $3\frac{1}{3}$, 3, $2\frac{8}{11}$, $2\frac{1}{2}$, &c.

Qu. 21. Two men carrying a burthen of 200 lb weight between them, hung on a pole, the ends of which rest on their shoulders; how much of this load is borne by each

each man, the weight hanging 6 inches from the middle, and the whole length of the pole being 4 feet ?

Anf. 125 lb and 75 lb.

Qu. 22. If, in a pair of scales, a body weigh 90 lb in one scale, and only 40 lb in the other ; required its true weight, and the proportion of the lengths of the two arms of the balance beam on each side of the point of suspension.

Anf. the weight 60 lb, and the propor. 3 to 2.

Qu. 23. To find the weight of a beam of timber, or other body, by means of a man's own weight, or any other weight. For instance, a piece of tapering timber, 24 feet long, being laid over a prop, or the edge of another beam, is found to balance itself when the prop is 13 feet from the left end ; but removing the prop a foot nearer to the said end, it takes a man's weight of 210 lb, standing on the left end, to hold it in equilibrium. Required the weight of the tree.

Anf. 2520 lb.

Qu. 24. If AB be a cane or walking-stick, 40 inches long, suspended by a string SD fastened to the middle point D : now a body being hung on at E, 6 inches distant from D, is balanced by a weight of 2 lb, hung on at the larger end A ; but removing the body to F, one inch nearer to D, the 2 lb weight on the other side is moved to G, within 8 inches of D, before the cane will rest in equilibrium. Required the weight of the body.

Anf. 24 lb.

Qu. 25. If AB, BC be two inclined planes, of the lengths of 30 and 40 inches, and moveable about the joint at B : what will be the ratio of two weights P, Q, in equilibrium upon the planes, in all positions of them ; and what will be the altitude BD of the angle B above the horizontal plane AC when this is 50 inches long ?

Anf. $BD = 24$; and P to Q as AB to BC, or as 3 to 4.

Qu. 26. A

Qu. 26. A lever of 6 feet long is fixed at right angles in a screw, whose threads are one inch asunder, so that the lever turns just once round in raising or depressing the screw one inch. If then this lever be urged by a weight or force of 50 lb, with what force will the screw press?

Anf. $22619\frac{1}{2}$ lb.

Qu. 27. If a man can draw a weight of 150 lb up the side of a perpendicular wall, of 20 feet high; what weight will he be able to raise along a smooth plank of 30 feet long, laid aslope from the top of the wall?

Anf. 225 lb.

Qu. 28. If a force of 150 lb be applied on the head of a rectangular wedge, its thickness being 2 inches, and the length of its side 12 inches; what weight will it raise or balance perpendicular to its side?

Anf. 900 lb.

Qu. 29. If a round pillar, of 30 feet diameter, be raised on a plane, inclined to the horizon in an angle of 75° , or the shaft inclining 15 degrees out of the perpendicular; what length will it bear before it overset?

Anf. $30(2 + \sqrt{3})$ or 111.9615 feet.

Qu. 30. If the greatest angle at which a bank of natural earth will stand, be 45° ; it is proposed to determine what thickness an upright wall of stone must be made throughout, just to support a bank of 12 feet high; the specific gravity of the stone being to that of earth, as 5 to 4.

Anf. $12\sqrt{\frac{8}{15}}$, or 8.76356 feet.

Qu. 31. If the stone wall be made like a wedge, or having its upright section a triangle, tapering to a point at top, but its side next the bank of earth perpendicular to

to the horizon; what is its thickness at the bottom so as to support the same bank?

Anf. $12\sqrt{\frac{4}{3}}$, or 10.733126 feet.

Qu. 32. But if the earth will only stand at an angle of 30 degrees to the horizontal line; it is required to determine the thickness of wall in both the preceding cases.

Anf. the breadths are the same as before, because the area of the triangular bank of earth is increased in the same proportion as its horizontal push is decreased.

Qu. 33. To find the thickness of an upright rectangular wall, necessary to support a body of water; the water being 10 feet deep, and the wall 12 feet high; also the specific gravity of the water, as 11 to 7. Anf. 4.204374 feet.

Qu. 34. To determine the thickness of the wall at the bottom, when the section of it is triangular, and the altitudes as before. Anf. 5.1492866 feet.

Qu. 35. Supposing the distance of the earth from the sun to be 95 millions of miles, I would know at what distance from him another body must be placed, so as to receive light and heat quadruple to that of the earth.

Anf. at half the distance, or $47\frac{1}{2}$ millions.

Qu. 36. If the mean distance of the sun from us be 106 of his diameters, how much hotter is it at the surface of the sun, than under our equator?

Anf. 11236 times hotter.

Qu. 37. The distance between the earth and sun being accounted 95 millions of miles, and between Jupiter and the sun 495 millions; the degree of light and heat received by Jupiter, compared with that of the earth, is required.

Anf. $\frac{361}{9801}$, or nearly $\frac{1}{27}$ of the earth's light and heat.

Qu. 38. A

Qu. 38. A certain body on the surface of the earth weighs a cwt, or 112 lb ; the question is whither this body must be carried, that it may weigh only 10 lb.

Anf. either at 3.3466 semi-diameters, or $\frac{5}{36}$ of a semi-diameter from the center.

Qu. 39. If a body weigh 1 pound or 16 ounces upon the surface of the earth, what will its weight be at 50 miles above it, taking the earth's diameter at 7930 miles.

Anf. 15 oz $9\frac{5}{8}$ dr nearly.

Qu. 40. Whereabouts, in the line between the earth and moon, is their common centre of gravity : supposing the earth's diameter to be 7930 miles, and the moon's 2160 ; also the density of the former to that of the latter, as 10 to 7, and their mean distance 30 of the earth's diameters ?

Anf. at $\frac{105}{251}$ parts of a diam, from the earth's center, or $\frac{61}{302}$ parts of a diameter, or 963 miles below the surface.

Qu. 41. Whereabouts, between the earth and moon, are their attractions equal to each other ? Or, where must another body be placed, so as to remain in equilibrio, not being more attracted to the one than to the other, or having no tendency to fall either way ? Their dimensions being as in the last question.

Anf. From the earth's center $26\frac{9}{11}$ } of the earth's di-
From the moon's centre $3\frac{2}{11}$ } ameters.

Qu. 42. Suppose a stone, dropt into an abyfs, should be stopped at the end of the 11th second after its delivery, what space would it have gone through ?

Anf. $1946\frac{1}{2}$ feet.

Qu. 43. What is the difference between the depths of two wells, into each of which should a stone be dropped

at the same instant, one will strike the bottom at 6 seconds, the other at 10? Anf. $1029\frac{1}{3}$ feet.

Qu. 44. If a stone be $19\frac{1}{2}$ seconds in descending from the top of a precipice to the bottom, what is its height? Anf. $6115\frac{1}{16}$ feet.

Qu. 45. In what time will a musket ball, dropped from the top of Salisbury steeple, said to be 400 feet high, reach the bottom? Anf. 5 sec. nearly.

Qu. 46. If a heavy body be observed to fall through 100 feet in the last second of time, from what height did it fall, and how long was it in motion? Anf. time $3\frac{2}{3}\frac{3}{8}\frac{5}{6}$ sec. and height $209\frac{4}{9}\frac{2}{2}\frac{7}{6}\frac{3}{4}$ feet.

Qu. 47. A stone being let fall into a well, it was observed that, after being dropped, it was 10 seconds before the sound of the fall at the bottom reached the ear. What is the depth of the well? Anf. 1270 feet nearly.

Qu. 48. It is proposed to determine the length of a pendulum vibrating seconds in the latitude of London, where a heavy body falls through $16\frac{1}{12}$ feet in the first second of time. Anf. 39.11 inches.

By experiment this length is found to be $39\frac{1}{8}$ inches.

Qu. 49. What is the length of a pendulum vibrating in 2 seconds; also in half a second, and in a quarter second?

Anf. the 2 second pendulum $156\frac{1}{2}$
the $\frac{1}{2}$ second pendulum $9\frac{2}{3}\frac{5}{2}$
the $\frac{1}{4}$ second pendulum $2\frac{5}{12}\frac{7}{8}$ inches.

Qu. 50. What difference will there be in the number of

of vibrations made by a pendulum of 6 inches long, and another of 12 inches long, in an hour's time?

Anf. $2692\frac{1}{2}$.

Qu. 51. Observed that while a stone was descending to measure the depth of a well, a string and plummet, that from the point of suspension, or the place where it was held, to the center of oscillation, measured just 18 inches, had made 8 vibrations. What was the depth of the well?

Anf. 412.61 feet.

Qu. 52. If a ball vibrate in the arch of a circle, 10 degrees on each side of the perpendicular; or a ball roll down the lowest 10 degrees of the arch; required the velocity at the lowest point: the radius of the circle, or length of the pendulum, being 20 inches.

Anf. 4.4213 feet per sec.

Qu. 53. If a ball descend down a smooth inclined plane, whose length is 100 feet, and altitude 10 feet; how long will it be in descending, and what will be the last velocity?

Anf. the veloc. 25.364 feet per sec. and time 7.8852 sec.

Qu. 54. If a cannon ball of 1 lb weight be fired against a pendulous block of wood, and striking the center of oscillation, cause it to vibrate an arc whose chord is 30 inches; the radius of that arc, or distance from the axis to the lowest point of the pendulum, being 118 inches, and the pendulum vibrating in small arcs 40 oscillations per minute. Required the velocity of the ball, and the velocity of the center of oscillation of the pendulum at the lowest point of the arc; the whole weight of the pendulum being 500 lb.

Anf. veloc. ball 1956.6054 feet per sec.
and veloc. cent. oscil. 3.9054 feet per sec.

Qu. 55. How

Qu. 55. How deep will a cube of oak sink in common water; each side of the cube being 1 foot?

Ans. $11\frac{1}{10}$ inches.

Qu. 56. How deep will a globe of oak sink in water; the diameter being 1 foot?

Ans. 9.9867 inches.

Qu. 57. If a cube of wood, floating in common water, have 3 inches of its height dry above the water, and $4\frac{8}{103}$ inches dry when in sea water; it is proposed to determine the magnitude of the cube, and what sort of wood it is made of.

Ans. the wood is oak, and each side 40 inches.

Qu. 58. An irregular piece of lead ore weighs in air 12 ounces, but in water only 7; and another fragment weighs in air $14\frac{1}{2}$ ounces, but in water only 9; required their comparative densities, or specific gravities.

Ans. as 145 to 132.

Qu. 59. An irregular fragment of glass in the scale weighs 171 grains, and another of magnet 102 grains; but in water the first fetches up no more than 120 grains, and the other 79: what then will their specific gravities turn out to be?

Ans. glass to magnet as 3933 to 5202, or nearly as 10 to 13.

Qu. 60. Hiero, king of Sicily, ordered his jeweller to make him a crown, containing 63 ounces of gold. The workmen thought that substituting part silver was only a proper perquisite; which taking air, Archimedes was appointed to examine it; who, on putting it into a vessel of water, found it raised the fluid 8.2245 cubic inches: and having discovered that the inch of gold more critically weighed 10.36 ounces, and that of silver but 5.85 ounces, he found

M

by

by calculation what part of his majesty's gold had been changed. And you are desired to repeat the process.

Anf. 28.8 ounces.

Qu. 61. Supposing the cubic inch of common glass weigh 1.4921 ounces avoirdupois, the same of sea water .59542, and of brandy .5368; then a seaman having a gallon of this liquor in a glass bottle, which weighs 3.84 lb out of water, and to conceal it from the officers of the customs, throws it overboard. It is proposed to determine, if it will sink, how much force will just buoy it up.

Anf. 14.1496 ounces.

Qu. 62. Another person has half an anker of brandy, of the same specific gravity as in the last question; the wood of the cask suppose measures $\frac{1}{8}$ of a cubic foot; it is proposed to assign what quantity of lead is just requisite to keep the cask and liquor under water.

Anf. 89.743 ounces.

Qu. 63. Suppose by measurement it be found that a man of war, with its ordnance, rigging, and appointments, sinks so deep as to displace 50000 cubic feet of water; what is the whole weight of the vessel?

Anf. $1395\frac{1}{10}$ tons.

Qu. 64. It is required to determine what would be the height of the atmosphere, if it were every where of the same density as at the surface of the earth, when the quicksilver in the barometer stands at 30 inches; and also what would be the height of a water barometer at the same time.

Anf. height of the air 175000 feet or 5.5240 miles,
height of water 35 feet.

Qu. 65. With what velocity would each of those
three

three fluids, viz, quicksilver, water, and air, issue through a small orifice in the bottom of vessels, of the respective heights of 30 inches, 35 feet, and 5.5240 miles; estimating the pressure by the half altitudes, and the air rushing into a vacuum? feet.

Ans. the veloc. of quicksilver 8.967

the veloc. of water 33.55

the veloc. of air 968.6

But estimating by the whole alt. the veloc. of air is 1369.8

And the mean between these two is 1169.2

which is nearly the velocity of sound, and also nearly equal to the velocity of a ball through the air when it suffers a resistance equal to the pressure of the atmosphere.

Qu. 66. A very large vessel of 10 feet high (no matter what shape) being kept constantly full of water, by a large supplying cock at the top; if 9 small circular holes, each $\frac{1}{5}$ of an inch diameter, be opened in its perpendicular side at every foot of the depth; it is required to determine the several distances to which they will spout on the horizontal plane of the base, and the quantity of water discharged by all of them in 10 minutes.

Ans. the distances are

$\sqrt{18}$ or 4.24264

$\sqrt{32}$ - 5.65685

$\sqrt{42}$ - 6.48074

$\sqrt{48}$ - 6.92820

$\sqrt{50}$ - 7.07106

$\sqrt{48}$ - 6.92820

$\sqrt{42}$ - 6.48074

$\sqrt{32}$ - 5.65685

$\sqrt{18}$ - 4.24264

and the quantity discharged in 10 min. 87.5997 gallons.

Qu. 67. If the inner axis of a hollow globe of copper, exhausted of air, be 100 feet; what thickness must it be of, that it may just float in air?

Anf. .02688 of an inch thick.

Qu. 68. If a spherical balloon of copper of $\frac{1}{100}$ of an inch thick, have its cavity of 100 feet diameter, and be filled with inflammable air of $\frac{1}{16}$ of the gravity of common air; what weight will just balance it, and prevent it from rising up into the atmosphere. Anf. 20453 lb.

Qu. 69. If a glass tube, 36 inches long, close at top, be sunk perpendicularly into water, till its lower or open end be 30 inches below the surface of the water; how high will the water rise within the tube, the quicksilver in the common barometer at the same time standing at $29\frac{1}{2}$ inches? Anf. 2.26545 inches.

Qu. 70. If a diving bell, of the form of a parabolic conoid, be let down into the sea to the several depths of 5, 10, 15, and 20 fathoms; it is required to assign the respective heights to which the water will rise within it: its axis and the diameter of its base being each 8 feet, and the quicksilver in the barometer standing at 30.9 inches.

Anf. at 5 fath. deep the water rises	2.03546 feet,
at 10	- - - 3.06393
at 15	- - - 3.70267
at 20	- - - 4.14658.

PRACTICAL QUESTIONS

To exercise the

DOCTRINE OF FLUXIONS.

QUEST. 1. **A** LARGE vessel of 10 feet high, and of any shape, being kept constantly full of water, by means of a supplying cock at the top; it is proposed to assign the place where a small hole must be made in the side of it, so that the water may spout through it to the greatest distance on the plane of the base.

Ans. the hole in the middle spouts farthest.

Qu. 2. If the same vessel stand on high, with its bottom a given height above a horizontal plane below; it is proposed to determine where the small hole must be, so as to spout farthest on the said plane.

Ans. in the middle between the plane and top of the vessel.

Qu. 3. But if the vessel stand on an inclined plane, making an angle of 30 degrees with the horizon, it is proposed to determine the place of the small hole, so as to spout the farthest on the said inclined plane.

Ans. $\frac{6 + \sqrt{6}}{10} a$ below the top, the altitude of the vessel being a .

Qu. 4. Required the size of a ball, which, being let fall into a conical glass full of water, shall expel the most water possible from the glass; its altitude being 6 inches, and diameter 5.

Anf. the diameter of the ball $4\frac{1}{6}$ inches.

Qu. 5. It is proposed to determine the altitude and diameter of a conical glass, capable of containing a pint of water, so that a heavy ball of 4 inches diameter, being dropt into it when full, may displace the most water possible.

Anf. the altitude 4.8450
and diameter 5.2716 inches.

Qu. 6. The distance between two horizontal planes is a , and they are both cut at an angle of 30 degrees by an oblique plane. Query from what point in the upper plane a ball must be dropped, so that striking the inclined plane, and thence rebounding from it, the ball may range the farthest possible on the lower horizontal plane; with the whole time the ball is in motion.

Anf. height above point struck on the obl. pl. $\frac{2 \mp \sqrt{3}}{3} a$,
the greatest range - - - - - a or $2a$,

the whole time $\sqrt{\frac{\frac{2}{3}a}{16\frac{1}{12}}} + \sqrt{\frac{\frac{2 \pm \sqrt{3}}{3}a}{16\frac{1}{12}}} = 4.8171$ when $a=100$.

Qu. 7. If an elastic ball be let fall from the height of 100 feet above the plane of the base of a hemisphere, of 10 feet diameter, lying upon a horizontal plane; it is proposed to determine on what part of the hemisphere the ball must impinge, so that it may thence rebound to the greatest distance on the plane; also to assign the extent of that distance, and the whole time in motion.

Anf. strikes the hemisf. at 4.0112 above the base,
the greatest distance 114.2846 without the hemisph.
and the whole time 4.16705" in motion.

Qu. 8. The

Qu. 8. The same hemisphere, of 10 feet diameter, lying on a horizontal plane, a ball begins to roll down it at $48' 36''$ from its vertex. Query the point at which the ball will quit the hemisphere, as also the place where it will strike the plane, and the whole time it is in motion.

Anf. quits at 3.333 above the base,
 the time $1.61662''$ on the hemisphere,
 the same 0.27465 after quitting it,
 therefore the whole time 1.89127 in motion,
 and strikes the plane at 1.62267 beyond the hemisphere.

PRACTICAL EXERCISES

CONCERNING FORCES; WITH THE RELATION BETWEEN
THEM AND THE TIME, VELOCITY, AND SPACE
DESCRIBED.

BEFORE entering upon the following problems, it will be convenient here to lay down a synopsis of the theorems which express the several relations between any forces, and their corresponding times, velocities, and spaces described; which are all comprehended in the following 12 theorems.

Let f , F be any two constant accelerative forces, acting on any body, during the respective times t , T , at the end of which are generated the velocities v , V , and described the spaces s , S . Then, because the spaces are as the times and velocities conjointly, and the velocities as the forces and times; we shall have,

I. *In Constant Forces,*

$$\begin{aligned} 1. \quad \frac{s}{S} &= \frac{tv}{TV} = \frac{t^2f}{T^2F} = \frac{v^2F}{V^2f}. \\ 2. \quad \frac{v}{V} &= \frac{ft}{FT} = \frac{sT}{St} = \sqrt{\frac{fS}{FS}}. \\ 3. \quad \frac{t}{T} &= \frac{Fv}{fV} = \frac{sV}{sV} = \sqrt{\frac{Fs}{fS}}. \\ 4. \quad \frac{f}{F} &= \frac{Tv}{tV} = \frac{T^2s}{t^2S} = \frac{v^2S}{V^2s}. \end{aligned}$$

And

And if one of the forces, as F , be the force of gravity at the surface of the earth, and be called 1 , and its time τ be $= 1''$; then it is known by experiment that the corresponding space s is $= 16\frac{1}{12}$ feet, and consequently its velocity $v = 2s = 32\frac{1}{6}$, which call $2g$, namely $g = 16\frac{1}{12}$ feet or 193 inches. Then the above four theorems, in this case, become as follows:

$$5. \quad s = \frac{tv}{2} = gft^2 = \frac{v^2}{4gf}.$$

$$6. \quad v = 2gft = \frac{2s}{t} = \sqrt{4gfs}.$$

$$7. \quad t = \frac{v}{2gf} = \frac{2s}{v} = \sqrt{\frac{s}{gf}}.$$

$$8. \quad f = \frac{v}{2gt} = \frac{s}{gt^2} = \frac{v^2}{4gs}.$$

And from these are deduced the following four theorems, for variable forces, viz.

II. In Variable Forces,

$$9. \quad \dot{s} = \frac{v\dot{v}}{2gf} = v\dot{t}.$$

$$10. \quad \dot{v} = \frac{2gf\dot{s}}{v} = 2gf\dot{t}.$$

$$11. \quad \dot{t} = \frac{\dot{s}}{v} = \frac{\dot{v}}{2gf}.$$

$$12. \quad f = \frac{v\dot{v}}{2gs} = \frac{\dot{v}}{2gt}.$$

In these four theorems, the force f , though variable, is supposed to be constant for the indefinitely small time $\dot{t}$; and they are to be used in all cases of variable forces, as the former ones in constant forces; namely, from the circumstances

stances of the problem under consideration, deduce an expression for the value of the force f , which substitute in one of these theorems, which shall be proper to the case in hand, and the equation thence resulting will determine the corresponding values of the other quantities required in the problem.

Note. That the motive force m , of a body, is equal to $f q$, the product of the accelerative force, and the quantity of matter in it; and, therefore, m will be found by substituting $\frac{m}{q}$ for f in the theorems above.

Also the momentum, or quantity of motion, is $q v$, the product of the velocity and matter.

It is also to be observed, that the theorems equally hold good for the destruction of motion and velocity, by means of retarding forces, as for the generation of the same by means of accelerating forces.

And to the following problems, which are all resolved by the application of these theorems, it has been thought proper to subjoin their solutions, for the better information and convenience of the student.

The following table of the most common and useful forms of fluxions and fluents, is inserted, as it will be found very useful for assigning the fluents found in many of the following problems.

Note. The logarithms mentioned in the fluents, are the hyperbolic ones, and the radius of the circle is 1.

Forms	Fluxions	Fluents
I	$x^{n-1} \dot{x}$	$\frac{1}{n} x^n$ { This form fails when $n = 0$
II	$\frac{a \pm x^n}{x^{m-1} \dot{x}}$	$\frac{\pm 1}{mn} \times a \pm x^n$ { Fails when m or $n = 0$
III	$\frac{x^{mn-1} \dot{x}}{a \pm x^n + 1}$	$\frac{1}{mna} \times \frac{x^{mn}}{a \pm x^n}$ { Fails when a or m or $n = 0$
IV	$x^{-1} \dot{x}$ or $\frac{x}{x}$	$\log. x$
V	$\frac{x^{n-1} \dot{x}}{a \pm x^n}$	$\pm \frac{1}{n} \log. a \pm x^n$ Fails when $n = 0$
VI	$\frac{x^{-1} \dot{x}}{a \pm x^n}$	$\frac{1}{an} \log. \frac{x^n}{a \pm x^n}$
VII	$\frac{x^{\frac{1}{2}n-1} \dot{x}}{a - x^n}$	$\frac{1}{n\sqrt{a}} \log. \frac{\sqrt{a} + \sqrt{x^n}}{\sqrt{a} - \sqrt{x^n}}$
VIII	$\frac{x^{\frac{1}{2}n-1} \dot{x}}{a + x^n}$	$\frac{1}{n\sqrt{a}} \times 2 \text{ arc tang. } \sqrt{\frac{x^n}{a}}$ or $\frac{1}{n\sqrt{a}} \times \text{arc cofin. } \frac{a - x^n}{a + x^n}$
IX	$\frac{x^{\frac{1}{2}n-1} \dot{x}}{\sqrt{\pm a + x^n}}$	$\frac{2}{n} \log. \sqrt{x^n} + \sqrt{\pm a + x^n}$
X	$\frac{x^{\frac{1}{2}n-1} \dot{x}}{\sqrt{a - x^n}}$	$\frac{1}{n} \times 2 \text{ arc fin. } \sqrt{\frac{x^n}{a}}$ or $\frac{1}{n} \times \text{arc vers. } \frac{2x^n}{a}$
XI	$\frac{x^{-1} \dot{x}}{\sqrt{a \pm bx^n}}$	$\frac{1}{n\sqrt{a}} \log. \frac{\pm \sqrt{a \pm x^n} \mp \sqrt{a}}{\sqrt{a \pm x^n} + \sqrt{a}}$
XII	$\frac{x^{-1} \dot{x}}{\sqrt{-a + x^n}}$	$\frac{1}{n\sqrt{a}} \times 2 \text{ arc sec. } \sqrt{\frac{x^n}{a}}$ or $\frac{1}{n\sqrt{a}} \times \text{arc cofin. } \frac{2a - x^n}{x^n}$

PROBLEM I.

To determine the time and velocity of a body descending, by the force of gravity, down an inclined plane; the length of the plane being 20 feet, and its height 1 foot.

Here, by mechanics, $20 : 1 :: 1$, the force of gravity:
 $\frac{1}{20} = f$, the force on the plane.

Theref. by theor. 6, v or $\sqrt{4gfs}$ is $\sqrt{4 \times 16\frac{1}{2} \times \frac{1}{20} \times 20}$
 $= \sqrt{4 \times 16\frac{1}{2}} = 2 \times 4\frac{1}{2} = 8\frac{1}{2}$ feet, the last velocity per second,

And, by theor. 7, t or $\sqrt{\frac{s}{gf}}$ is $\sqrt{\frac{20}{16\frac{1}{2} \times \frac{1}{20}}} = \sqrt{\frac{400}{16\frac{1}{2}}}$
 $= \frac{20}{4\frac{1}{2}} = 4\frac{7}{7}$ seconds, the time of descending.

PROBLEM II.

If a cannon ball be fired with a velocity of 1000 feet per second, in the direction of, and up, a smooth inclined plane, that arises 1 foot in 20: it is proposed to assign the length which it will ascend up the plane before it stops and begins to return down again, and the time of its ascent.

Here $f = \frac{1}{20}$ as before.

Then, by theor. 5, $s = \frac{v^2}{4gf} = \frac{1000^2}{4 \times 16\frac{1}{2} \times \frac{1}{20}} =$
 $\frac{60000000}{193} = 310880\frac{160}{193}$ feet, or nearly 59 miles, the distance moved.

And, by theor. 7, $t = \frac{v}{2gf} = \frac{1000}{2 \times 16\frac{1}{2} \times \frac{1}{20}} = \frac{120000}{193}$
 $= 621''\frac{147}{193} = 10' 21''\frac{147}{193}$, the time of ascent.

PROBLEM

PROBLEM III.

If a ball be projected up a smooth inclined plane, which rises 1 foot in 10, and ascend 100 feet before it stop: required the time of ascent, and the velocity of projection.

First, by theor. 6, $v = \sqrt{4gs} = \sqrt{4 \times 16\frac{1}{2} \times \frac{1}{10} \times 20} = 8\frac{1}{4} \sqrt{10} = 25.36408$ feet per second, the velocity.

And, by theor. 7, $t = \sqrt{\frac{s}{gf}} = \sqrt{\frac{100}{16\frac{1}{2} \times \frac{1}{10}}} = \frac{10}{4\frac{1}{2}} \sqrt{10} = 1\frac{22}{77} \sqrt{10} = 7.88516$ seconds, the time in motion.

PROBLEM IV.

If a ball be observed to ascend up a smooth inclined plane, 100 feet in ten seconds, before it stop, to return back again: required the velocity of projection, and the angle of the plane's inclination.

First, by theor. 6, $v = \frac{2s}{t} = \frac{200}{10} = 20$ feet per sec. the velocity.

And, by theor. 8, $f = \frac{s}{gt^2} = \frac{100}{16\frac{1}{2} \times 100} = \frac{12}{193}$. That

is, the length of the plane is to its height, as 193 to 12.

Therefore, $193 : 12 :: 100 : 6.2176$ the height of the plane, or the sine of elevation to radius 100, which answers to $3^\circ 34'$, the angle of elevation of the plane.

PROBLEM V.

By a mean of several experiments, I have found that a cast iron ball, of 2 inches diameter, impinging perpendicularly on the face or end of a block of elm wood, or in the direction of the fibres, with a velocity of 1500 feet per second, penetrated 13 inches deep into its substance. It is proposed

proposed then to determine the time of the penetration, and the resisting force of the wood as compared to the force of gravity, supposing that force to be a constant quantity.

First, by theor. 7, $t = \frac{2s}{v} = \frac{2 \times 13}{1500 \times 12} = \frac{1}{692}$ part of a second, the time.

And, by th. 8, $f = \frac{v^2}{4gs} = \frac{1500^2}{4 \times 16\frac{1}{2} \times \frac{13}{12}} = \frac{81000000}{13 \times 193} = 32284$. That is, the resisting force of the wood, is to the force of gravity, as 32284 to 1.

But this number will be different, according to the diameter of the ball; and its density or specific gravity.

For, since f is as $\frac{v^2}{s}$ by theor. 4, the density and size of

the ball remaining the same; if the density, or specific gravity, n , vary, and all the rest be constant, it is evident that f will be as n ; and therefore f as $\frac{nv^2}{s}$ when the size

of the ball only is constant. But when only the diameter d varies, all the rest being constant, the force of the blow will vary as d^3 ; or as the magnitude of the ball; and the resisting surface, or force of resistance, varies as d^2 ; therefore f is as $\frac{d^3}{d^2}$, or as d only when all the rest are con-

stant. Consequently f is as $\frac{dnv^2}{s}$ when they are all variable.

$$\text{And so } \frac{f}{F} = \frac{dnv^2s}{DNV^2s}, \text{ and } \frac{s}{S} = \frac{dnv^2F}{DNV^2f};$$

where f denotes the strength or firmness of the substance penetrated, and is here supposed to be the same, for all balls and velocities, in the same substance, which it is either accurately or nearly so. See pa. 264 &c. of my Tracts, vol. 1.

Hence,

Hence, taking the numbers in the problem, it is

$$f = \frac{dnv^2}{s} = \frac{\frac{2}{12} \times 7\frac{1}{3} \times 1500^2}{\frac{13}{12}} = \frac{44 \times 1500^2}{39} = 2538462$$

the value of f for elm wood. Where the specific gravity of the ball is taken $7\frac{1}{3}$, which is a little less than that of solid cast iron, as it ought, on account of the air bubble that is cast in all balls.

PROBLEM VI.

To find how far a 24 lb ball of cast iron will penetrate into a block of sound elm, when fired with a velocity of 1600 feet per second.

Here, because the substance is the same as in the last problem, both of the balls and wood, $N = n$, and $F = f$;

therefore $\frac{s}{s} = \frac{DV^2}{dv^2}$, or $s = \frac{DV^2s}{dv^2} = \frac{5.55 \times 1600^2 \times 13}{2 \times 1500^2}$
 $= 41\frac{1}{2}$ inches nearly, the penetration required.

PROBLEM VII.

It was found by Mr. Robins (vol. 1 pa. 273) that an 18 pounder ball, fired with a velocity of 1200 feet per second, penetrated 34 inches into sound dry oak. It is required then to ascertain the comparative strength or firmness of oak and elm.

The diameter of an 18 lb ball is 5.04 inches = D . Then, by the numbers given in this problem for oak, and in prob.

v for elm, we have $\frac{f}{F} = \frac{dv^2s}{DV^2s} = \frac{2 \times 1500^2 \times 34}{5.04 \times 1200^2 \times 13}$
 $= \frac{100 \times 17}{5.04 \times 16 \times 13} = \frac{1700}{1048}$ or $= \frac{7}{5}$ nearly.

From which it would seem that elm resists more than oak, in the ratio of about 7 to 5; which is not probable, as oak is a much firmer and harder wood. But it is to

be suspected that Mr. Robins's great penetration was owing to the splitting of his timber in some degree.

PROBLEM VIII.

A 24 pounder ball being fired into a bank of firm earth, with a velocity of 1300 feet per second, penetrated 15 feet: It is required then to ascertain the comparative resistances of elm and earth.

Comparing the numbers here with those in prob. 5, it is $\frac{f}{F} = \frac{dv^2s}{DV^2S} = \frac{2 \times 1500^2 \times 15 \times 12}{5.55 \times 1300^2 \times 13} = \frac{15^2 \times 24}{13^2 \times 0.37} = \frac{1800}{271} = \frac{20}{3}$ nearly $= 6\frac{2}{3}$ nearly. That is, elm resists about $6\frac{2}{3}$ times more than earth.

PROBLEM IX.

To determine how far a leaden bullet, of $\frac{3}{4}$ of an inch diameter, will penetrate dry elm; supposing it fired with a velocity of 1700 feet per second, and that the lead does not change its figure by the stroke against the wood.

Here $D = \frac{3}{4}$, $N = 11\frac{1}{3}$, $n = 7\frac{1}{3}$. Then by the numbers and theorem in prob. 5, it is $s = \frac{DNV^2s}{dnv^2} = \frac{\frac{3}{4} \times 11\frac{1}{3} \times 1700^2 \times 13}{2 \times 7\frac{1}{3} \times 1500^2} = \frac{17^3 \times 13}{200 \times 33} = \frac{63869}{6600} = 9\frac{2}{3}$ inches nearly, the depth of penetration.

But as Mr. Robins found this penetration, by experiment, to be only 5 inches; it follows, either that his timber must have resisted about twice as much; or else, which is very probable, that the defect in his penetration arose from the change of figure in the leaden ball from the blow against the wood.

PROBLEM X.

A one pound ball, projected with a velocity of 1500 feet per second, having been found to penetrate 13 inches deep into dry elm: It is required to ascertain the time of passing through every single inch of the 13, and the velocity lost at each of them; supposing the resistance of the wood constant or uniform.

The velocity v being 1500 feet, or $1500 \times 12 = 9000$ inches, and velocities and times being as the roots of the spaces, in constant retarding forces, as well as in accelerating ones, and t being $= \frac{2s}{v} = \frac{26}{12 \times 1500} = \frac{13}{9000} = \frac{1}{692}$ part of a second, the whole time of passing through the 13 inches; therefore as

$$\sqrt{13} : \sqrt{13} - \sqrt{12} :: v :$$

veloc. loft

Time In the

$\frac{\sqrt{13} - \sqrt{12}}{\sqrt{13}} v = 58.8 :: t :$	$\frac{\sqrt{13} - \sqrt{12}}{\sqrt{13}} t = .00005$	1st
$\frac{\sqrt{12} - \sqrt{11}}{\sqrt{13}} v = 61.4 :: t :$	$\frac{\sqrt{12} - \sqrt{11}}{\sqrt{13}} t = .00006$	2d
$\frac{\sqrt{11} - \sqrt{10}}{\sqrt{13}} v = 64.2$	&c $\frac{\sqrt{11} - \sqrt{10}}{\sqrt{13}} t = .00006$	3d
$\frac{\sqrt{10} - \sqrt{9}}{\sqrt{13}} v = 67.5$	$\frac{\sqrt{10} - \sqrt{9}}{\sqrt{13}} t = .00007$	4th
$\frac{\sqrt{9} - \sqrt{8}}{\sqrt{13}} v = 71.4$	$\frac{\sqrt{9} - \sqrt{8}}{\sqrt{13}} t = .00007$	5th
$\frac{\sqrt{8} - \sqrt{7}}{\sqrt{13}} v = 76.0$	$\frac{\sqrt{8} - \sqrt{7}}{\sqrt{13}} t = .00007$	6th
$\frac{\sqrt{7} - \sqrt{6}}{\sqrt{13}} v = 81.6$	$\frac{\sqrt{7} - \sqrt{6}}{\sqrt{13}} t = .00008$	7th
$\frac{\sqrt{6} - \sqrt{5}}{\sqrt{13}} v = 88.8$	$\frac{\sqrt{6} - \sqrt{5}}{\sqrt{13}} t = .00008$	8th
$\frac{\sqrt{5} - \sqrt{4}}{\sqrt{13}} v = 98.3$	$\frac{\sqrt{5} - \sqrt{4}}{\sqrt{13}} t = .00009$	9th
$\frac{\sqrt{4} - \sqrt{3}}{\sqrt{13}} v = 111.4$	$\frac{\sqrt{4} - \sqrt{3}}{\sqrt{13}} t = .00011$	10th
$\frac{\sqrt{3} - \sqrt{2}}{\sqrt{13}} v = 132.3$	$\frac{\sqrt{3} - \sqrt{2}}{\sqrt{13}} t = .00013$	11th
$\frac{\sqrt{2} - \sqrt{1}}{\sqrt{13}} v = 172.3$	$\frac{\sqrt{2} - \sqrt{1}}{\sqrt{13}} t = .00017$	12th
$\frac{\sqrt{1} - \sqrt{0}}{\sqrt{13}} v = 416.0$	$\frac{\sqrt{1} - \sqrt{0}}{\sqrt{13}} t = .00040$	13th
Sum	<u>1500.0</u>	Sum $\frac{1}{692}$ or <u>.00144</u> inch

Hence, as the motion loft at the beginning is very small; and consequently the motion communicated to any body, as an inch plank, in passing through it, is very small also; we can conceive such a plank shot through, without oversetting it;

it; and may easily compute the height and breadth of such a plank as will just stand without being overset by the ball in passing through it.

PROBLEM XI.

To determine the circumstances of space, penetration, velocity, and time, arising from a ball moving with a given velocity, and striking a moveable block of wood, or other substance.



Let the ball move in the direction AE passing through the centre of gravity of the block B, impinging on the point c; and when the block has moved through the space CD, in consequence of the blow, let the ball have penetrated to the depth DE.

Let B = the mass or matter in the block,

b = the same in the ball,

s = CD the space moved by the block,

x = DE the penetration of the ball, and theref.

$s + x$ = CE the space described by the ball,

a = the first velocity of the ball,

v = the velocity of the ball at E,

u = veloc. of the block at the same instant,

t = the time of penetration, or of the motion,

r = the resisting force of the wood.

Then shall $\frac{r}{B}$ be the accelerating force of the block,

and $\frac{r}{b}$ the retarding force of the ball.

Now because the momentum $B\dot{u}$, communicated to the block in the time t , is that which is lost by the ball, namely $-b\dot{v}$, therefore $B\dot{u} = -b\dot{v}$, and $Bu = -bv$. But when $v = a$, $u = a$; therefore, by correcting, $Bu = b(a - v)$; or the momentum of the block is every where equal to the momentum lost by the ball. And when the ball has penetrated to the utmost depth, or when $u = v$, this becomes $Bu = b(a - u)$, or $ab = (B + b)u$; that is, the momentum before the stroke, is equal to the momentum after it. And the velocity communicated will be the same, whatever be the resisting force of the block, the weight being the same.

Again, by theor. 6, it is $u^2 = \frac{4grs}{B}$, and $-v^2 = \frac{4gr}{b} \times (s + x)$, or rather, by correction, $a^2 - v^2 = \frac{4gr}{b} (s + x)$.

Hence the penetration or $x = \frac{b(a^2 - v^2) - 4grs}{4gr}$. And when $v = u$, by substituting u for v , and Bu^2 for $4grs$, the greatest penetration becomes $\frac{ba^2 - (B + b)u^2}{4gr}$; and this again, by writing ab for its value $(B + b)u$, gives the greatest penetration $x = \frac{Bba^2}{4gr(B + b)} = \frac{ba^2}{4gr} \times (1 - \frac{b}{B + b})$. Which is barely equal to $\frac{ba^2}{4gr}$ when the block is fixed, or infinitely great; and is always very nearly equal to the same $\frac{ba^2}{4gr}$ when B is very great in respect of b . Hence

$$s + x = \frac{a^2 - u^2}{4gr} = \frac{a^2 - \frac{a^2b^2}{(B + b)^2}}{4gr} = \frac{B^2 + 2Bb}{(B + b)^2} \times \frac{a^2b}{4gr}.$$

And therefore $B + b : B + 2b :: x : s + x$,

or $B + b : b :: x : s$,

$$\text{and } s = \frac{bx}{B + b} = \frac{Bb^2a^2}{4gr(B + b)^2}.$$

Ex.

Ex. When the ball is iron, and weighs 1 pound, it penetrates elm about 13 inches when it moves with a velocity of 1500 feet per second, in which case,

$$\frac{r}{b} = \frac{a^2}{4gx} = \frac{1500^2}{4 \times 16\frac{1}{2} \times \frac{13}{12}} = \frac{9000^2}{193 \times 13} = 32284$$

nearly.

When $B = 500$ lb, and $b = 1$; then $u = \frac{ab}{B + b} = \frac{1500}{501} = 3$ feet nearly per second, the velocity of the block.

Also $s = \frac{Bu^2}{4gr} = \frac{500 \times 9}{4 \times 16\frac{1}{2} \times 32284} = \frac{1}{461\frac{1}{2}}$
 part of a foot, or $\frac{2}{77}$ of an inch, which is the space moved by the block when the ball has completed its penetration.

And $t = \frac{2s}{u} = \frac{2}{461\frac{1}{2} \times 3} = \frac{1}{692}$ part of a second,
 or $t = \frac{2s + 2x}{v} = \frac{\frac{2}{461\frac{1}{2}} + \frac{26}{12}}{1500} = \frac{6 + 13231}{6.231.1500} = \frac{1}{692}$
 part of a second, the time of penetration.

For the circumstances relating to the motion of a block, hung by, and vibrating on, an axis, when struck by a ball, see my Tracts, pa. 116, &c.

PROBLEM XII.

The force of attraction, above the earth, being inversely as the square of the distance from the centre; it is proposed to determine the time, velocity, and other circumstances, attending a heavy body falling from any given height; the descent at the earth's surface being $16\frac{1}{2}$ feet, or 193 inches, in the first second of time.

Put

$r =$ cs the radius of the earth,

$a =$ CA the dist. fallen from,

$x =$ CP any variable distance,

$v =$ the velocity at P,

$t =$ time of falling there, and

$g = 16\frac{1}{12}$, half the veloc. or force at A,

$f =$ the force at the point P.



Then we have the three following equations,

$$x^2 : r^2 :: 2g : f = \frac{2gr^2}{x^2} \text{ the force at P, or the velocity}$$

per second that would be generated by the force there,

$$tv = -\dot{x}, \text{ and}$$

$$vv = -f\dot{x} = -\frac{2gr^2\dot{x}}{x^2}.$$

The fluents of the last equation give $v^2 = \frac{4gr^2}{x}$.

But when $x = a$, the velocity $v = 0$; therefore, by cor-

rection, $v^2 = \frac{4gr^2}{x} - \frac{4gr^2}{a} = 4gr^2 \times \frac{a-x}{ax}$; or $v =$

$\sqrt{\frac{4gr^2}{a} \times \frac{a-x}{x}}$, a general expression for the velocity at any point P.

When $x = r$, this gives $v = \sqrt{4gr \times \frac{a-r}{r}}$ for the greatest velocity, or the velocity when the body strikes the earth.

When a is very great in respect of r , the last velocity becomes $(1 - \frac{r}{2a}) \times \sqrt{4gr}$ very nearly, or nearly $\sqrt{4gr}$ only, which is accurately the greatest velocity by falling from an infinite height. And this, when $r = 3965$ miles, is 6.9506 miles per second. Also the velocity acquired in falling from the distance of the sun, or 12000 diameters of the earth, is 6.9505 miles per second. And the velocity acquired in falling from the distance of the moon, or 30 diameters, is 6.8924 miles per second.

Again,

Again, to find the time, since $\dot{v} = -\dot{x}$, therefore

$$\dot{x} = \frac{-\dot{x}}{v} = \sqrt{\frac{a}{4gr^2}} \times \frac{-x\dot{x}}{\sqrt{ax - xx}};$$
 the correct fluent
of which gives $t = \sqrt{\frac{a}{4gr^2}} \times (\sqrt{ax - xx} + \text{arc to di-}$
 $\text{ameter } a \text{ and vers. } a - x);$ or the time of falling to any
point $P = \frac{1}{2r} \sqrt{\frac{a}{g}} \times (AB + BP).$ And when $x = r$,
this becomes $t = \frac{1}{2} \sqrt{\frac{a}{g}} \times \frac{AD + DS}{sc}$ for the whole
time of falling to the surface at s ; which is evidently in-
finite when a or AC is infinite, although the velocity is
then only the finite quantity $\sqrt{4gr}.$

When the height above the earth's surface is given $= g$;
because r is then nearly $= a$, and AD nearly $= DS$, the
time t for the distance g will be nearly $\sqrt{\frac{r}{4gr^2}} \times 2DS$
 $= \sqrt{\frac{1}{4gr}} \times \sqrt{4gr} = 1'',$ as it ought to be.

If a body at the distance of the moon at A fall to the
earth's surface at s . Then $r = 3965$ miles, $a = 60r$,
and $t = 416806'' = 4\text{da. } 19\text{h. } 46' 46'',$ the time of falling
from the moon to the earth.

When the attracting body is considered as a point c ;
the whole time of descending to c will be

$$\frac{1}{2r} \sqrt{\frac{a}{g}} \times ABDC = \frac{.7854a}{r} \sqrt{\frac{a}{g}}.$$

PROBLEM XIII.

The force of attraction below the earth's surface being directly as the distance from the center; it is proposed to determine the circumstances of velocity, time, and space fallen by a heavy body from the surface, through a perforation made straight to the centre of the earth; abstracting from the effect of the earth's rotation, and supposing it to be a homogeneous sphere of 3965 miles radius.

Put $r = AC$ the radius of the earth,

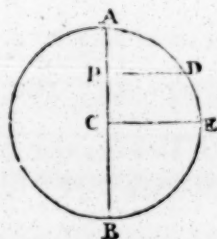
$x = CP$ the distance fallen,

$v =$ the velocity at P ,

$t =$ the time there,

$g = 16\frac{1}{12}$, half the force at A ,

$f =$ the force at P .



Then $CA : CP :: 2g : f$; and the three equations are $rf = 2gx$, and $v\dot{v} = -fx$, and $\dot{v} = -\dot{x}$.

Hence $f = \frac{2gx}{r}$, and $v\dot{v} = -\frac{gxx}{r}$; the correct fluent of

which gives $v = \sqrt{2g \times \frac{r^2 - x^2}{r}} = PD\sqrt{\frac{2g}{r}} = PD\sqrt{\frac{2g}{CE}}$

the velocity at the point P ; where PD and CE are perpendicular to CA . So that the velocity at any point P , is as the perpendicular PD at that point.

When the body arrives at c , then $v = \sqrt{2gr} = \sqrt{2g \cdot AC} = 25950$ feet or 4.9148 miles per second, which is the greatest velocity, or that at the centre c .

Again, for the time, $\dot{t} = \frac{-\dot{x}}{v} = \sqrt{\frac{r}{2g}} \times \frac{-\dot{x}}{\sqrt{r^2 - x^2}}$,

and the fluents give $t = \sqrt{\frac{r}{2g}} \times \text{arc to cosine } \frac{x}{r} =$

$\sqrt{\frac{r}{2gr}} \times \text{arc } AD$. So that the time of descent to any point P , is as the corresponding arc AD .

When

When P arrives at c , the above becomes $t = \sqrt{\frac{r}{2g}}$
 $\times$ quadrant $AE = \frac{AE}{AC} \sqrt{\frac{r}{2g}} = 1.5708 \sqrt{\frac{r}{2g}} = 1267\frac{1}{4}$
 seconds $= 21' 7''\frac{1}{4}$, for the time of falling to the centre c .

The time of falling to the centre is the same quantity $1.5708 \sqrt{\frac{r}{2g}}$, from whatever point in the radius AC the body begins to move. For let n be any given distance from c at which the motion commences : then, by correction,

$$v = \sqrt{\frac{2g}{r} (n^2 - x^2)}; \text{ and hence } \dot{t} = \sqrt{\frac{r}{2g}} \times \frac{-\dot{x}}{\sqrt{n^2 - x^2}},$$

the fluents of which give $t = \sqrt{\frac{r}{2g}} \times \text{arc to cosine}$

$\frac{x}{n}$; which, when $x = 0$, gives $t = \sqrt{\frac{r}{2g}} \times \text{quadrant} =$

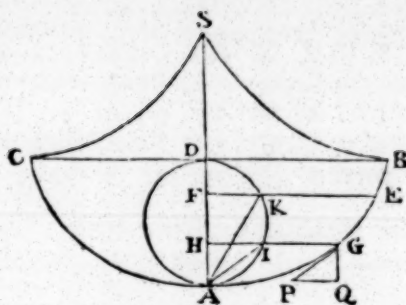
$1.5708 \sqrt{\frac{r}{2g}}$ for the time of descent to the centre c .

As an equal force, acting in contrary directions, generates or destroys an equal quantity of motion, in the same time; it follows that, after passing the centre, the body will just ascend to the opposite surface at B , in the same time in which it fell to the centre from A ; then from B it will return again in the same manner, through c to A ; and so vibrate continually between A and B , the velocity being always equal at equal distances from c on both sides; and the whole time of a double oscillation, or of passing from A and arriving at A again, will be quadruple the time of passing over the radius AC , or =

$$2 \times 3.1416 \sqrt{\frac{r}{2g}} = 1^h 24' 29''.$$

PROBLEM XIV.

To find the Time of a Pendulum vibrating in the Arc of a Cycloid.



Let s be the point of suspension,

SA = the arc SB or SC the length of the pendulum,

$CA = AB = SB$ or SC the semi-cycloid,

$AD = DS$ the diameter of its generating circle, to which FKE , HIG are perpendiculars.

To any point G draw the tangent GP , also draw GQ parallel and PQ perpendicular to AD . Then PG is parallel to the chord AI by the nature of the curve. And, by the nature of forces, the force of gravity : force in direct. $GP :: GP : GQ :: AI : AH :: AD : AI$; in like manner, the force of grav. : force in curve at $E :: AD : AK$; that is, the accelerative force in the curve, is as the corresponding chord AI or AK of the circle, or as the arc AG or AE of the cycloid, since AG is always $= 2AI$. So that the process and conclusions for the velocity and time of describing any arc in this case, will be the same as in the last problem.

From whence it follows that the time of a semi-vibration, in all arcs, AG , AE , &c, is the same constant quantity

tity $1.5708\sqrt{\frac{r}{2g}} = 1.5708\sqrt{\frac{A}{2g}} = 1.5708\sqrt{\frac{l}{2g}}$, and the time of a whole vibration from B to C, or from C to B, is $3.1416\sqrt{\frac{l}{2g}}$, where $l = AS = AB$ is the length of the pendulum, $g = 16\frac{1}{12}$ feet or 193 inches, and 3.1416 the circumference of a circle whose diameter is 1.

Since the time of a body's falling by gravity through $\frac{1}{2}l$, or half the length of the pendulum, is $\sqrt{\frac{l}{2g}}$, which being in proportion to $3.1416\sqrt{\frac{l}{2g}}$, as 1 to 3.1416; therefore the diameter of a circle is to its circumference, as the time of falling through half the length of a pendulum, to the time of one vibration.

If the time of the whole vibration be 1 second, this equation arises, $1'' = 3.1416\sqrt{\frac{l}{2g}}$, and hence $l = \frac{2g}{3.1416^2} = \frac{g}{4.9348}$, and $g = 3.1416^2 \times \frac{1}{2}l = 4.9348l$. So that if one of these, g or l , be given by experiment, these equations will give the other. When g , for instance, is supposed to be $16\frac{1}{12}$ feet, or 193 inches, then is $l = \frac{g}{4.9348} = 39.11$ the length of a pendulum to vibrate seconds. Or if $l = 39\frac{1}{8}$, the length of the seconds pendulum for the latitude of London, then is $g = 4.9348l = 193.07$ inches $= 16\frac{1.07}{12.00}$ feet, or nearly $16\frac{1}{12}$ feet, for the space descended by gravity in the first second of time in the latitude of London.

Hence the times of vibration of pendulums, are as the square roots of their lengths; and the number of vibrations made in a given time, reciprocally as the square roots of the lengths. And hence the length of a pendulum

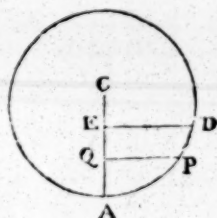
vibrating n times in a minute, or $60''$, is $l = 39\frac{1}{8} \times \frac{60^2}{n^2} = \frac{140850}{nn}$.

When a pendulum vibrates in a circular arc, as the length of the string is constantly the same, the time of vibration will be longer than in a cycloid; but the two times will approach nearer together as the circular arc is smaller; so that when it is very small, the times of vibration will be nearly equal. And hence $39\frac{1}{8}$ inches is the length of a pendulum vibrating seconds in the very small arc of a circle.

PROBLEM XV.

To find the Velocity and Time of a Heavy Body descending down the Arc of a Circle, or vibrating in the Arc by a Line fixed in the Centre.

Let D be the beginning of the descent, C the centre, and A the lowest point of the circle; draw DE and PQ perpendicular to AC . Then the velocity in P being the same as in Q by falling through EQ , it will be $v =$



$2\sqrt{g \times EQ} = 2\sqrt{g(a-x)}$, when $a = AE$, $x = AQ$, and $g = 16\frac{1}{12}$.

But the fluxion of the time t is $= \frac{-\dot{AP}}{v}$, and $\dot{AP} =$

$\frac{r\dot{x}}{\sqrt{2rx-x^2}}$, where r = the radius AC . Therefore

$$\dot{t} = \frac{r}{2\sqrt{g}} \times \frac{-\dot{x}}{\sqrt{2rx-x^2} \times \sqrt{a-x}} = \frac{d}{4\sqrt{g}} \times \frac{-\dot{x}}{\sqrt{ax-x^2} \times \sqrt{d-x}}, \text{ where } d = 2r \text{ the diameter.}$$

Or

$$\text{Or } \dot{t} = \frac{1}{4} \sqrt{\frac{d}{g}} \times \frac{-\dot{x}}{\sqrt{ax - x^2}} \left(1 + \frac{x}{2d} + \frac{1 \cdot 3x^2}{2 \cdot 4d^2} + \frac{1 \cdot 3 \cdot 5x^3}{2 \cdot 4 \cdot 6d^3} \&c. \right).$$

But the fluent of $\frac{\dot{x}}{\sqrt{ax - x^2}}$ is $\frac{2}{a} \times$ arc to rad. $\frac{1}{2}a$ and vers. x , or it is the arc whose rad. is 1 and vers. $\frac{2x}{a}$: which call A. And let the fluents of the succeeding terms, without the coefficients, be B, C, D, E, &c. Then will the fluxion of any one, as $\dot{Q}$, at n distance from A, be $\dot{Q} = x^n \dot{A} = x \dot{P}$, which suppose also = the fluxion of $bP - dx^{n-1} \sqrt{ax - x^2} = b\dot{P} - d \cdot n - 1 \cdot x x^{n-2} \sqrt{ax - x^2} - dx^{n-2} \times \frac{\frac{1}{2}ax - x^2}{\sqrt{ax - x^2}} = b\dot{P} - dx \times \frac{n - \frac{1}{2} \cdot ax^{n-1} - nx^n}{\sqrt{ax - x^2}} = b\dot{P} - d \cdot n - \frac{1}{2} \cdot a\dot{P} + dnx\dot{P}.$

Hence, by equating the coefficients of the like terms, $d = \frac{1}{n}$; $b = \frac{2n-1}{2n}a$; and $Q = \frac{2n-1 \cdot aP - 2x^{n-1} \sqrt{ax - x^2}}{2n}.$

Which being substituted, the fluential terms become

$$\frac{1}{4} \sqrt{\frac{d}{g}} \times \left(-A - \frac{1}{2d} \frac{aA - 2\sqrt{ax - x^2}}{2} - \frac{1 \cdot 3}{2 \cdot 4d^2} \cdot \frac{3aB - 2x\sqrt{ax - x^2}}{4} - \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6d^3} \cdot \frac{5aC - 2x^2\sqrt{ax - x^2}}{6} - \&c. \right).$$

But when $x = a$, those terms become barely

$$\frac{3 \cdot 1416}{4} \sqrt{\frac{d}{g}} \times \left(-1 - \frac{1^2 a}{2^2 d} - \frac{1^2 \cdot 3^2 a^2}{2^2 \cdot 4^2 d^2} - \frac{1^2 \cdot 3^2 \cdot 5^2 a^3}{2^2 \cdot 4^2 \cdot 6^2 d^3} - \&c. \right);$$

which being subtracted, and x taken = 0, there arises for the whole time of descending down DA, or the corrected value of $t = \frac{3 \cdot 1416}{4} \sqrt{\frac{d}{g}} \times \left(1 + \frac{1^2 a}{2^2 d} + \frac{1^2 \cdot 3^2 a^2}{2^2 \cdot 4^2 d^2} + \frac{1^2 \cdot 3^2 \cdot 5^2 a^3}{2^2 \cdot 4^2 \cdot 6^2 d^3} + \&c. \right).$

When

When the arc is small, as in the vibration of the pendulum of a clock, all the terms of the series may be omitted after the second, and then the time of a vibration t is nearly $= 1.5708 \sqrt{\frac{r}{2g}} \times (1 + \frac{a}{8r})$. And therefore the times of vibration of a pendulum, in different arcs, are as $8r + a$, or 8 times the radius added to the versed sine of the arc.

If D be the degrees of the pendulum's vibration, on each side of the lowest point of the small arc, the radius being r , the diameter d , and $3.1416 = p$; then is the length of that arc $A = \frac{p r D}{180}$. But the versed sine in terms of the arc is $a = \frac{A^2}{2r} - \frac{A^4}{24r^3} + \&c = \frac{A^2}{d} - \frac{A^4}{3d^3} + \&c$. Therefore $\frac{a}{d} = \frac{A^2}{d^2} - \frac{A^4}{3d^4} + \&c = \frac{p^2 D^2}{360^2} - \frac{p^4 D^4}{360^4} + \&c$, or only $= \frac{p^2 D^2}{360^2}$ the first term, by rejecting all the rest of the terms on account of their smallness, or $\frac{a}{d} = \frac{a}{2r}$ nearly $= \frac{D^2}{12787}$. This value then being substituted for $\frac{a}{d}$ or $\frac{a}{2r}$ in the last near value of the time, it becomes $t = 1.5708 \sqrt{\frac{r}{2g}} \times (1 + \frac{D^2}{51150})$ nearly. And therefore the times of vibration in different small arcs, are as $51150 + D^2$, or as 51150 added to the square of the number of degrees in the arc.

Hence it follows that the time lost in each second, by vibrating in a circle, instead of the cycloid, is $\frac{D^2}{51150}$; and consequently the time lost in a whole day of 24 hours, or $24 \times 60 \times 60$ seconds, is $\frac{5}{3} D^2$ nearly. In like manner, the seconds lost per day by vibrating in the arc of Δ degrees, is $\frac{5}{3} \Delta^2$. Therefore, if the pendulum keep true

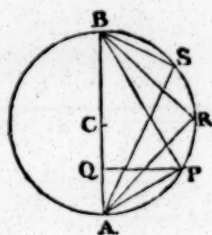
true time in one of these arcs, the seconds lost or gained per day, by vibrating in the other, will be $\frac{5}{3} (D^2 - \Delta^2)$. So, for example, if a pendulum measure true time in an arc of 3 degrees, it will lose $11\frac{2}{3}$ seconds a day by vibrating 4 degrees; and $26\frac{2}{3}$ seconds a day by vibrating 5 degrees; and so on.

And in like manner, we might proceed for any other curve, as the ellipse, hyperbola, parabola, &c.

PROBLEM XVI.

To determine the Time of a Body descending down the Chord of a Circle.

Let c be the centre, AB the vertical diameter, AP any chord down which a body is to descend from P to A , and PQ perpendicular to AB . Now as the natural force of gravity in the vertical direction BA , is to the force urging the body down the plane PA , as the length of the plane AP , is to its height AQ ; therefore the velocity in PA and QA , will be equal at all equal perpendicular distances below PQ ; and consequently the time in PA : time in QA :: PA : QA :: BA : PA ; but time in BA : time in QA :: $\sqrt{BA}$: $\sqrt{QA}$:: BA : PA ; hence, as three of the terms in each proportion are the same, the fourth terms must be equal, namely the time in BA = the time PA .



And in like manner the time in BP = the time in BA . So that, in general, the times of descending down all the chords BA , BP , BR , BS , &c, or PA , RA , SA , &c, are all equal, and each equal to the time of falling freely through
the

the diameter. Which time is $\sqrt{\frac{2r}{g}}$, where $g = 16\frac{1}{2}$ feet, and $r =$ the radius AC; for $\sqrt{g} : \sqrt{2r} :: 1'' : \sqrt{\frac{2r}{g}}$.

SCHOLIUM. By comparing this with the results of the two preceding problems, it will appear that the times in the cycloid, and in the arc of a circle, and in any chord of the circle, are respectively as the three quantities

$$1, 1 + \frac{a}{8r} \text{ \&c, and } \frac{1}{.7854}$$

or nearly as the three quantities $1, 1 + \frac{a}{8r}, 1.27324$; the first and last being constant, but the middle one, or the time in the circle, varying with the extent of the arc of vibration. Also the time in the cycloid is the least, but in the chord the greatest; for the greatest value of the series, in prob. 15, when $a = r$, or the arc AD is a quadrant, is 1.18014 ; and in that case the proportion of the three times is as the numbers $1, 1.18014, 1.27324$. Moreover the time in the circle approaches to that in the cycloid, as the arc decreases, and they are very nearly equal when that arc is very small.

PROBLEM XVII.

To find the Time and Velocity of a Chain, consisting of very small links, descending from a smooth horizontal plane; the Chain being 100 inches long, and 1 Inch of it hanging off the Plane at the Commencement of Motion.

Put $a = 1$ inch, the length at the beginning;
 $l = 100$ the whole length of the chain;
 $x =$ any variable length off the plane.

Then x is the motive force to move the body,
 and $\frac{x}{l} = f$ the accelerative force.

$$\text{Hence } v\dot{v} = 2gf\dot{s} = 2g \times \frac{x}{l} \times \dot{x} = \frac{2gx\dot{x}}{l}.$$

The fluents give $v^2 = \frac{2gx^2}{l}$. But $v = 0$ when $x = a$, therefore, by correction, $v^2 = 2g \times \frac{x^2 - a^2}{l}$,
 and $v = \sqrt{2g \times \frac{x^2 - a^2}{l}}$ the velocity for any length x . And when the chain just quits the plain, $x = l$,
 and then the greatest velocity is $\sqrt{2g \times \frac{l^2 - a^2}{l}} =$
 $\sqrt{2 \times 193 \times \frac{100^2 - 1^2}{100}} = \sqrt{\frac{386 \times 9999}{100}} = 196.45902$
 inches, or 16.371585 feet, per second.

Again $\dot{t}$ or $\frac{\dot{s}}{v} = \sqrt{\frac{l}{2g}} \times \frac{\dot{x}}{\sqrt{x^2 - a^2}}$; the correct fluent
 of which is $t = \sqrt{\frac{l}{2g}} \times \log. \frac{x + \sqrt{x^2 - a^2}}{a}$, the time
 for any length x . And when $x = l = 100$, it is $t =$

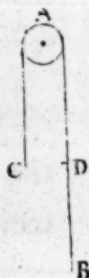
$\sqrt{\frac{100}{386}} \times \log. \frac{100 + \sqrt{9999}}{1} = 2.69676$ seconds,
the time when the last of the chain just quits the plane.

PROBLEM XVIII.

To find the Time and Velocity of a Chain, of very small Links, quitting a Pulley, by passing freely over it: the whole Length being 200 Inches, and the one End hanging 2 Inches below the other at the Beginning.

Put $a = 2$, $l = 200$, and $x = BD$ any variable difference of the two parts AB , AC .

Then $\frac{x}{l} = f$, and $v\dot{v}$ or $2gf\dot{s} = 2g \cdot \frac{x}{l} \cdot \frac{1}{2}\dot{x} = \frac{gxx\dot{x}}{l}$.



Hence the correct fluent is $v^2 = g \times \frac{x^2 - a^2}{l}$, and

$v = \sqrt{g \times \frac{x^2 - a^2}{l}}$, the general expression of the velocity. And when $x = l$, or c arrives at A , it is
 $v = \sqrt{g \times \frac{l^2 - a^2}{l}} = \sqrt{193 \times \frac{200^2 - 2^2}{200}} = \sqrt{386}$
 $\times \frac{100^2 - 1^2}{200} = \sqrt{\frac{386 \times 9999}{100}} = 196.45902$ inches, or
 16.371585 feet, for the greatest velocity when the chain just quits the pulley.

Again, $\dot{t}$ or $\frac{\dot{s}}{v} = \frac{\dot{x}}{2v} = \sqrt{\frac{l}{4g}} \times \frac{\dot{x}}{\sqrt{x^2 - a^2}}$. And

the correct fluent is $t = \sqrt{\frac{l}{4g}} \times \log. \frac{x + \sqrt{x^2 - a^2}}{a}$,
the general expression for the time. And when $x = l$,
it becomes $t = \sqrt{\frac{l}{4g}} \times \log. \frac{l + \sqrt{l^2 - a^2}}{a} = \sqrt{\frac{200}{772}} \times \log.$

$$\times \log. \frac{200 + \sqrt{200^2 - 2^2}}{2} = \sqrt{\frac{100}{386}} \times 1. \frac{100 + \sqrt{9999}}{2}$$

= 2.69676 seconds, the whole time when the chain just quits the pulley.

So that the velocity and time at quitting the pulley in this prob. and the plane in the last prob. are the same; the distance descended 99 being the same in both. For, although the weight l moved in this latter case, be double of what it was in the former, the moving force x is also double, because here the one end of the chain shortens as much as the other end lengthens, so that the space descended $\frac{1}{2}x$ is doubled, and becomes x ; and hence the accelerative force $\frac{x}{l}$ or f is the same in both; and of course the velocity and time the same for the same distance descended.

PROBLEM XIX.

To find the Number of Vibrations made by two Weights, connected by a very fine Thread, passing freely over a Tack or a Pulley, while the less Weight is drawn up to it by the Descent of the heavier Weight at the other End.

Suppose the motion to commence at equal distances below the pulley at B; and that the weights are 1 and 2 pounds.

Put $a = AB$, half the length of the thread;
 $b = 39\frac{1}{8}$ inc. or $3\frac{2}{9}\frac{5}{8}$ feet, the second's pend.
 $x = Bw = BW$, any space passed over;
 $z =$ the number of vibrations:

Then $\frac{w - w}{w + w} = f = \frac{1}{3}$ is the accelerating force. And,



hence v or $\sqrt{4gfs} = \sqrt{4gfx}$, and i or $\frac{\dot{x}}{v} = \frac{\dot{x}}{\sqrt{4gfx}}$.

But, by the nature of pendulums, $\sqrt{a \pm x} : \sqrt{b} :: 1 \text{ vibr.} :$

$\sqrt{\frac{b}{a \pm x}}$ the vibrations per second made by either weight, namely, the longer or shorter, according as the upper or under sign is used, if the threads were to continue of that length for 1 second. Hence, then, as

$$x'' : i :: \sqrt{\frac{b}{a \pm x}} : \dot{x} = i \sqrt{\frac{b}{a \pm x}} = \sqrt{\frac{b}{4gf}} \times \frac{\dot{x}}{\sqrt{ax \pm x^2}},$$

the fluxion of the number of vibrations.

Now when the upper sign $+$ takes place, the fluent is

$$z = 2\sqrt{\frac{b}{4gf}} \times \log. \frac{\sqrt{x} + \sqrt{a+x}}{\sqrt{a}} = \sqrt{\frac{b}{4gf}} \times \log. \frac{a + 2x + 2\sqrt{ax + x^2}}{a}.$$

And when $x = a$, the same

$$\text{then becomes } z = \sqrt{\frac{b}{gf}} \times \log. 1 + \sqrt{2} = \sqrt{\frac{3b}{g}} \times \log. 1 + \sqrt{2} = \sqrt{\frac{117\frac{3}{8}}{193}} \times \log. 1 + \sqrt{2} = .688511,$$

the whole number of vibrations made by the descending weight.

But when the lower sign, or $-$, takes place, the fluent is $\sqrt{\frac{b}{4gf}} \times \text{arc to rad. } 1 \text{ and vers. } \frac{2x}{a}$. Which, when

$$x = a, \text{ gives } \frac{1}{2}p\sqrt{\frac{b}{gf}} = 3.1416 \times \sqrt{\frac{3 \times 39\frac{1}{8}}{4 \times 193}} = \frac{3.1416}{2} \times \sqrt{\frac{117\frac{3}{8}}{193}} = 1.227091, \text{ the whole number of vibrations made by the lesser or ascending weight.}$$

SCHOL. It is evident that the whole number of vibrations, in each case, is the same, whatever the length

of the thread is. And that the greater number is to the less, as 1.5708 to hyp. log. $1 + \sqrt{2}$.

Farther, the number of vibrations performed in the same time t , by an invariable pendulum, constantly of the same length a , is $\sqrt{\frac{b}{gf}} = .781190$. For the time of descending the space a , or the fluent of $\dot{x} = \frac{x}{\sqrt{4gfx}}$, when $x = a$, is $t = \sqrt{\frac{a}{gf}}$. And, by the nature of pendulums, $\sqrt{a} : \sqrt{b} :: 1 \text{ vibr.} : \sqrt{\frac{b}{a}}$ the number of vibrations performed in 1 second; hence $1'' : t :: \sqrt{\frac{b}{a}} : t\sqrt{\frac{b}{a}} = \sqrt{\frac{b}{gf}}$, the constant number of vibrations.

So that the three numbers of vibrations, namely, of the ascending, constant, and descending pendulums, are proportional to the numbers 1.5708, 1, and hyp. log. $1 + \sqrt{2}$, or as 1.5708, 1, and .88137; whatever be the length of the thread.

PROBLEM XX.

To determine the Circumstances of the Ascent and Descent of two unequal Weights, suspended at the two Ends of a Thread passing over a Pulley: the Weight of the Thread and of the Pulley being considered in the Solution. Let

l = the whole length of the thread;
 a = the weight of the same;
 b = Aw the dif. of lengths at first;
 d = $w - w$ the dif. of the two weights;
 c = a wt. applied to the circumference, such as to be equal to its whole wt. and friction reduced to the circumference;

s = $w + w + a + c$ the sum of the weights moved.



Then the weight of b is $\frac{ab}{l}$, and $d - \frac{ab}{l}$ is the moving force at first. But if x denote any variable space descended by w , or ascended by w , the difference of the lengths of the thread will be altered $2x$; so that the difference will then be $b - 2x$, and its weight $\frac{b - 2x}{l} a$; consequently the motive force there will be $d - \frac{b - 2x}{l} a$
 $= \frac{dl - ab + 2ax}{l}$, and theref. $\frac{dl - ab + 2ax}{sl} = f$ the accelerating force there. Hence then $v\dot{v} = 2gf\dot{x} = 2gx \times \frac{dl - ab + 2ax}{sl}$; the fluents of which give
 $v^2 = 4gx \times \frac{dl - ab + 2ax}{sl}$, or $v = 2\sqrt{\frac{ag}{sl}} \times \sqrt{ex + x^2}$
the

the general expression for the velocity, putting $e = \frac{dl - ab}{a}$. And when $x = b$, or w becomes as far below w as it was above it at the beginning, it is barely $v = 2\sqrt{\frac{bdg}{s}}$ for the velocity at that time. Also, when a , the weight of the thread, is nothing, the velocity is only $2\sqrt{\frac{dgs}{s}}$, as it ought.

Again, for the time, t or $\frac{\dot{x}}{v} = \frac{1}{2}\sqrt{\frac{sl}{ag}} \times \frac{\dot{x}}{\sqrt{ex + x^2}}$;

the fluents of which give $t = \sqrt{\frac{sl}{ag}} \times \log. \frac{\sqrt{x + \sqrt{e + x}}}{\sqrt{e}}$

the general expression for the time of descending any space x .

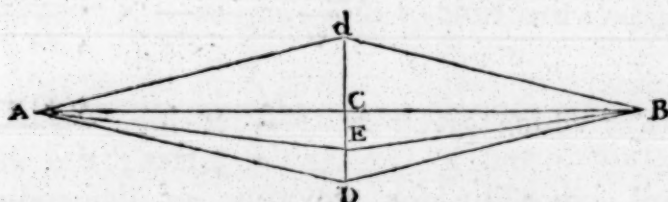
And if the radicals be expanded in a series, and the log. of it be taken, the same time will become

$t = \sqrt{\frac{sx}{dg}} \times \sqrt{\frac{dl}{dl - ab}} \times (1 - \frac{x}{6e} + \frac{3x^2}{40e^2} \&c)$. Which,

therefore, becomes barely $\sqrt{\frac{sx}{dg}}$ when a , the weight of the thread, is nothing; as it ought.

PROBLEM XXI.

To find the Velocity and Time of Vibration of a small Weight, fixed to the middle of a Line, or fine Thread void of Gravity, and stretched by a given Tension; the Extent of the Vibration being very small,



Let $l = AC$ half the length of the thread;
 $a = CD$ the extent of the vibration;
 $x = CE$ any variable distance from c ;
 $w = \text{wt. of the small body fixed to the middle}$;
 $W = \text{a wt. which, hung at each end of the thread,}$
 will be equal to the constant tension at each
 end, acting in the direction of the thread.

Now, by the nature of forces, $AE : CE :: w$ the force in direction EA : the force in direction EC . Or, because AC is nearly $= AE$, the vibration being very small, taking AC instead of AE , it is $AC : CE :: w : \frac{wx}{l}$ the force in EC arising from the tension in EA . Which will be also the same for that in EB . Therefore the sum is $\frac{2wx}{l} =$ the whole motive force in EC arising from the tensions on both sides. Consequently $\frac{2wx}{lw} = f$ the accelerative force there. Hence the equation of the fluxions is

is $v\dot{v}$ or $2gf\dot{s} = \frac{-4gWx\dot{x}}{lw}$; and the fluents $v^2 = -\frac{4gWx^2}{lw}$. But when $x = a$, this is $-\frac{4gWa^2}{lw}$, and should be $= 0$; therefore the correct fluents are $v^2 = 4gW \times \frac{a^2 - x^2}{lw}$, and $v = \sqrt{4gW \times \frac{a^2 - x^2}{lw}}$ the velocity of the little body w at any point E . And when $x = 0$, it is $v = 2a\sqrt{\frac{gW}{lw}}$ for the greatest velocity at the point c .

Now if we suppose $w = 1$ grain, $W = 5$ lb troy, or 28800 grains, and $2l = AB = 3$ feet; the velocity c becomes $a\sqrt{\frac{8 \times 16\frac{1}{2} \times 28800}{3}} = 1111\frac{2}{5}a$. So that

if $a = \frac{1}{10}$ inc. the greatest veloc. is $9\frac{1}{4}$ ft. per sec.

if $a = 1$ inc. the greatest veloc. is $92\frac{37}{50}$ ft. per sec.

if $a = 6$ inc. the greatest veloc. is $555\frac{7}{10}$ ft. per sec.

To find the time t , it is $\dot{t}$ or $\frac{-\dot{x}}{v} = \frac{1}{2}\sqrt{\frac{lw}{Wg}} \times \frac{-\dot{x}}{\sqrt{a^2 - x^2}}$.

Hence the correct fluent is $t = \frac{1}{2}\sqrt{\frac{wl}{Wg}} \times \text{arc to cosine}$

$\frac{x}{a}$ and radius 1, the time in DE . And when $x = 0$, the

whole time in DC , or of half a vibration, is $.7854\sqrt{\frac{wl}{Wg}}$;

and consequently the time of a whole vibration through Dd is $1.5708\sqrt{\frac{wl}{Wg}}$.

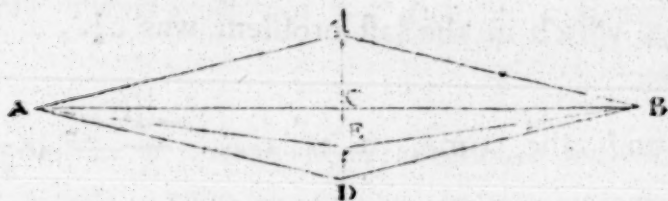
Using the foregoing numbers, namely $w = 1$, $W = 28800$, and $2l = 3$ feet; this expression for the time is

$\frac{1111\frac{2}{5}}{3.1416} = 353\frac{5}{9}$, the number of vibrations per second.

But if $w = 2$, there would be 250 vibrations per second; and if $w = 100$, there would be $35\frac{16}{45}$ vibrations per second.

PROBLEM XXII.

To determine the same as in the last Problem, when the Distance CD bears some sensible Proportion to the Length AB ; the Tension of the Thread however being still supposed a Constant Quantity.



Using here the same notation as in the last problem, and taking the true variable length AE for AC , it is AE or

$EB : CE :: 2W : \frac{2Wx}{AE} = \frac{2Wx}{\sqrt{l^2 + x^2}}$ the whole motive force from the two equal tensions w in AE and EB ; and therefore $\frac{2W}{w} \times \frac{x}{\sqrt{l^2 + x^2}} = f$ is the accelerative force at E .

Therefore the fluxional equation is $v\dot{v}$ or $2gf\dot{s} = \frac{4Wg}{w} \times \frac{-x\dot{x}}{\sqrt{l^2 + x^2}}$; and the fluents $v^2 = \frac{8Wg}{w} \times$

$-\sqrt{l^2 + x^2}$. But when $x = a$, these are 0 = $\frac{8Wg}{w} \times$

$-\sqrt{l^2 + a^2}$; therefore the correct fluents are $v^2 = \frac{8Wg}{w}$

$\times (\sqrt{l^2 + a^2} - \sqrt{l^2 + x^2}) = \frac{8Wg}{w} \times (AD - AE)$. And

hence $v = \sqrt{\frac{8Wg}{w} \times (AD - AE)}$ the general expression for the velocity at E . And when E arrives at C , it

gives the greatest velocity there = $\sqrt{\frac{8Wg}{w} \times (AD - AC)}$.

Which,

Which, when $w = 28800$, $w = 1$, $2l = 3$ feet, and $cd = 6$ inches or $\frac{1}{2}$ a foot, is $\sqrt{8 \times 28800 \times 16 \frac{1}{12} \times \frac{\sqrt{10} - 3}{2}}$
 $= 548 \frac{1}{3}$ feet per second. Which came out $555 \frac{7}{10}$ in the last problem, by using always AC for AE in the value of f . But when the extent of the vibrations is very small, as $\frac{1}{10}$ of an inch, as it commonly is, this greatest velocity here will be $\sqrt{8 \times 28800 \times 16 \frac{1}{12} \times \frac{1}{43200}} = 9 \frac{1}{2}$ nearly, which in the last problem was $9 \frac{1}{4}$.

To find the time, it is $\dot{i}$ or $\frac{-\dot{x}}{v} = \sqrt{\frac{w}{8wg}} \times \frac{-\dot{x}}{\sqrt{c - \sqrt{l^2 + x^2}}}$, making $c = AD = \sqrt{l^2 + a^2}$.—To find the fluent the easier, multiply the numerator and denominator both by $\sqrt{c + \sqrt{l^2 + x^2}}$, so shall $\dot{i} = \sqrt{\frac{w}{8wg}} \times \frac{-\dot{x}}{\sqrt{a^2 - x^2}} \times \sqrt{c + \sqrt{l^2 + x^2}}$. Expand now the quantity $\sqrt{c + \sqrt{l^2 + x^2}}$ in a series, and put $d = c + l$, so shall $\dot{i} = \sqrt{\frac{wd}{8wg}} \times (1 + \frac{x^2}{4dl} - \frac{2d+l}{32d^2l^3}x^4 + \frac{4d^2+2dl+l^2}{128d^3l^5}x^6 - \frac{40d^3+8d^2l+12dl^2+5l^3}{2048d^4l^7}x^8 \&c.)$

Now the fluent of the first term $\frac{-\dot{x}}{\sqrt{a^2 - x^2}}$ is = the arc to sine $\frac{x}{a}$ and radius 1, which arc call A ; and let P, Q be the fluents of any other two successive terms, without the coefficients, the distance of Q from the first term A being n ; then it is evident that $\dot{Q} = x^2 \dot{P} = x^{2n} \dot{A}$, and $\dot{P} = x^{2n-2} \dot{A}$. Assume therefore $Q = bP - ex^{2n-1} \sqrt{a^2 - x^2}$; then is $\dot{Q}$ or $x^2 \dot{P} = b \dot{P} - \frac{2n-1}{2n-1} e x^{2n-2} \dot{x} \sqrt{a^2 - x^2}$

$$\begin{aligned} \sqrt{a^2 - x^2} + \frac{ex^{2n}\dot{x}}{\sqrt{a^2 - x^2}} &= b\dot{P} - \frac{(2n-1)ea^2x^{2n-2}\dot{x}}{\sqrt{a^2 - x^2}} \\ + \frac{(2n-1)ex^{2n}\dot{x}}{\sqrt{a^2 - x^2}} + \frac{ex^{2n}\dot{x}}{\sqrt{a^2 - x^2}} &= b\dot{P} - (2n-1)ea^2\dot{P} \\ + (2n-1)ex^2\dot{P} + ex^2\dot{P} &= b\dot{P} - (2n-1)ea^2\dot{P} + \\ 2nex^2\dot{P}. \end{aligned}$$

Then, comparing the coefficients of the like terms, we find $1 = 2en$, and $b = (2n-1)ea^2$; from which are obtained $e = \frac{1}{2n}$, and $b = \frac{2n-1}{2n}a^2$. Con-

sequently $Q = \frac{(2n-1)a^2P - x^{2n-1}\sqrt{a^2 - x^2}}{2n}$, the general equation between any two successive terms, and by means of which the series may be continued as far as we please. And hence, neglecting the coefficients, putting

A = the first term, namely the arc whose sine is $\frac{x}{a}$, and $B, C, D, \&c$, the following terms, the series is as follows,

$$\begin{aligned} A + \frac{a^2A - \sqrt{xa^2 - x^2}}{2} + \frac{3a^2B - x^3\sqrt{a^2 - x^2}}{4} + \\ \frac{5a^2C - x^5\sqrt{a^2 - x^2}}{6} \&c. \end{aligned}$$

Now when $x = 0$, this series = 0; and when $x = a$, the series becomes $\frac{1}{2}p + \frac{a^2A}{2} + \frac{3a^2B}{4} + \frac{5a^2C}{6} \&c$, where $p = 3.1416$, or the series is $\frac{1}{2}p (1 + \frac{1}{2}a^2 + \frac{1 \cdot 3}{2 \cdot 4}a^4 + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6}a^6 \&c.)$

So that, by taking in the coefficients, the general time of passing over any distance DE will be $\sqrt{\frac{w(c+l)}{8wg}} \times$

$$\begin{aligned} \frac{1}{2}p \times (1 + \frac{1}{4dl} \cdot \frac{1}{2}a^2 - \frac{2d+l}{32d^2l^3} \cdot \frac{1 \cdot 3}{2 \cdot 4}a^4 \&c, \\ - \text{arc fin.} \frac{x}{a} - \frac{1}{4dl} \frac{a^2A - x\sqrt{a^2 - x^2}}{2} + \frac{2d+l}{32d^2l^3} \frac{3a^2B - x^3\sqrt{a^2 - x^2}}{4} \\ \&c. \end{aligned}$$

And

And hence, taking $x = 0$, and doubling, the time of a whole vibration, or double the time of passing over CD

will be equal to $\frac{1}{2}p\sqrt{\frac{w(c+l)}{2wg}} \times (1 + \frac{1}{4dl} \cdot \frac{1}{2}a^2 - \frac{2d+l}{32d^2l^3} \cdot \frac{1 \cdot 3}{2 \cdot 4} a^4 + \frac{4d^2+2dl+l^2}{128d^3l^5} \cdot \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6} a^6 - \frac{40d^3+8d^2l+12dl^2+5l^3}{2048d^4l^7} \cdot \frac{1 \cdot 3 \cdot 5 \cdot 7}{2 \cdot 4 \cdot 6 \cdot 8} a^8 \&c.)$ Which,

when $a = 0$, or $c = l$, becomes only $\frac{1}{2}p\sqrt{\frac{wl}{wg}}$, the same as in the last problem, as it ought.

Taking here the same numbers as in the last problem, viz. $l = \frac{3}{2}$, $a = \frac{1}{2}$, $w = 2$, $W = 28800$, $g = 16\frac{1}{12}$;

then $\frac{1}{2}p\sqrt{\frac{w(c+l)}{2wg}} = .0040514$, and the series is

$1 + .006762 - .000175 + .000003 \&c = 1.006590$;

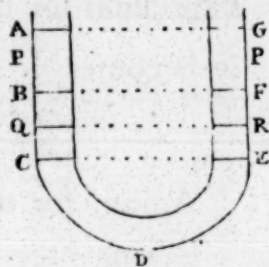
therefore $.0040514 \times 1.006590 = .0040965 = \frac{1}{245\frac{1}{5}}$

is the time of one whole vibration, and consequently $245\frac{1}{5}$ vibrations are performed in a second: which were 250 in the last problem.

PROBLEM XXIII.

It is proposed to determine the Velocity, and the Time of Vibration, of a Fluid in the Arms of a Canal or bent Tube.

Let the tube ABCDEF have its two branches AC, GE vertical, and the lower part CDE in any position whatever, the whole being of a uniform diameter or width throughout. Let water, or quicksilver, or any other fluid, be poured in, till it stand in equilibrio, at any horizontal line BF. Then let one surface be pressed or pushed down by shaking, from B to c, and the other will ascend through the equal space FG; after which let them be permitted freely to return. The surfaces will then continually vibrate in equal times between AC and EG. The velocity and times of which oscillations are therefore required.



When the surfaces are any where out of a horizontal line, as at P and Q, the parts of the fluid in QPR, on each side, below QR, will balance each other; and the weight of the part in PR, which is equal to 2PF, gives motion to the whole. So that the weight of the part 2PF is the motive force by which the whole fluid is urged, and there-

fore $\frac{\text{wt of } 2PF}{\text{whole wt.}}$ is the accelerative force. Which

weights being proportional to their lengths, if l be the length of the whole fluid, or axis of the tube filled, and $a = FG$ or BC ; then is $\frac{a}{l}$ the accelerative force. Putting therefore $x = GP$ any variable distance, v the velocity,

velocity, and t the time; then $PF = a - x$, and $\frac{2a - 2x}{l} = f$ the accelerative force; hence $v\dot{v}$ or $2gf\dot{s}$
 $= \frac{4g}{l} (a\dot{x} - x\dot{x})$; the fluents of which give $v^2 = \frac{4g}{l}$
 $(2ax - x^2)$, and $v = \sqrt{4g \times \frac{2ax - x^2}{l}}$ is the general
 expreffion for the velocity at any term. And when $x = a$,
 it becomes $v = 2a\sqrt{\frac{g}{l}}$ for the greatest velocity at B and F.

Again, for the time, we have $\dot{t}$ or $\frac{\dot{s}}{v} = \frac{1}{2}\sqrt{\frac{l}{g}} \times$
 $\frac{\dot{x}}{\sqrt{2ax - x^2}}$; the fluents of which give $t = \frac{1}{2}\sqrt{\frac{l}{g}} \times$
 arc to verfed sine $\frac{x}{a}$ and radius 1, the general expref-
 fion for the time. And when $x = a$, it becomes $t = \frac{1}{4}p$
 $\sqrt{\frac{l}{g}}$ for the time of moving from G to F, p being
 $= 3.1416$; and confequently $\frac{1}{2}p\sqrt{\frac{l}{g}}$ the time of a whole
 vibration from G to E, or from C to A. And which
 therefore is the fame, whatever AB is, the whole length l
 remaining the fame.

And the time of vibration is alfo equal to the time of
 the vibration of a pendulum whofe length is $\frac{1}{2}l$, or half
 the length of the axis of the fluid. So that if the length
 l be $78\frac{1}{4}$ inches, it will oscillate in 1 fecond.

SCHOL. This reciprocation of the water in the canal,
 is nearly fimilar to the motion of the waves of the fea.
 For the time of vibration is the fame, however fhort the
 branches are, provided the whole length be the fame. So
 that when the height is fmall, in proportion to the length
 of

of the canal, the motion is similar to that of a wave, from the top to the bottom or hollow, and from the bottom to the top of the next wave; being equal to two vibrations of the canal; the whole length of a wave, from top to top, being double the length of the canal. Hence the wave will move forward by a space nearly equal to its breadth, in the time of two vibrations of a pendulum whose length is $(\frac{1}{2}l)$ half the length of the canal, or one fourth of the breadth of a wave, or in the time of one vibration of a pendulum whose length is the whole breadth of the wave, since the times of vibration are as the square roots of their lengths. Consequently, waves whose breadth is equal to $39\frac{1}{8}$ inches, or $3\frac{2}{9}\frac{5}{8}$ feet, will move over $3\frac{2}{9}\frac{5}{8}$ feet in a second, or $195\frac{5}{8}$ feet in a minute, or nearly 2 miles and a quarter in an hour. And the velocity of greater or less waves will be increased or diminished in the subduplicate ratio of their breadths.

PROBLEM XXIV.

To determine the Time of filling the Ditches of a Work with Water at the Top, by a Sluice of 2 Feet square; the Head of Water above the Sluice being 10 Feet, and the Dimensions of the Ditch being 20 Feet wide at Bottom, 22 at Top, 9 deep, and 1000 Feet long.

The capacity of the ditch is 189000 cubic feet.

But $\sqrt{g} : \sqrt{10} :: 2g : 2\sqrt{10g}$ the velocity of the water through the sluice, the area of which is 4 square feet; therefore $8\sqrt{10g}$ is the quantity per second running through it; and consequently $8\sqrt{10g} : 189000 :: 1'' : \frac{23625}{\sqrt{10g}} = 1863''$ or $31' 3''$ nearly is the time filling the ditch.

PROBLEM XXV.

To determine the Time of emptying a Vessel of Water by a Sluice in the Bottom of it, or in the Side near the Bottom, the Height of the Aperture being very small in respect of the Altitude of the Fluid.

Put a = the area of the aperture or sluice ;

$2g = 32\frac{1}{6}$ feet, the force of gravity ;

d = the whole depth of water ;

x = the variable alt. of the surface above the aperture ;

A = the area of the surface of the water.

Then $\sqrt{g} : \sqrt{x} :: 2g : 2\sqrt{gx}$ the velocity with which the fluid will issue at the sluice ; and hence $A : a :: 2\sqrt{gx} : \frac{2a\sqrt{gx}}{A}$ the velocity with which the surface of the water will descend at the altitude x , or the space it would descend in 1 second with the velocity there. Now in descending the space x , the velocity may be considered as uniform ; and uniform descents are as their times ; therefore $\frac{2a\sqrt{gx}}{A} : \dot{x} :: 1'' : \frac{A\dot{x}}{2a\sqrt{gx}}$ the time of descending $\dot{x}$ space, or the fluxion of the time of exhausting. That is,

$$\dot{t} = \frac{-A\dot{x}}{2a\sqrt{gx}}.$$

Then when the nature or figure of the vessel is given, there will be given A in terms of x ; which value of A being substituted into this fluxion of the time, the fluent of the result will be the time of exhausting sought.

So if, for example, the vessel be any prism, or every where of the same breadth ; then A is a constant quantity,

P tity,

tity, and therefore the fluent is $-\frac{A}{a}\sqrt{\frac{x}{g}}$. But when $x = d$, this becomes $-\frac{A}{a}\sqrt{\frac{d}{g}}$, and should be 0; therefore the correct fluent is $t = \frac{A}{a} \times \frac{\sqrt{d} - \sqrt{x}}{\sqrt{g}}$ for the time of the surface descending till the depth of the water be x . And when $x = 0$, the whole time of exhausting is barely $\frac{A}{a}\sqrt{\frac{d}{g}}$.

And hence if A be 10000 square feet, $a = 1$ square foot, and $d = 10$ feet; the time is $7885\frac{1}{3}$ seconds, or $2^h 11' 25''\frac{1}{3}$.

Again, if the vessel be a ditch, or canal, of 20 feet broad at the bottom, 22 at the top, 9 deep, and 1000 feet long; then is $99 : 90 + x :: 22 : \frac{90 + x}{99} \times 22$ the breadth of the surface of the water when its depth in the canal is x ; and consequently $A = \frac{90 + x}{99} \times 22000$ is the surface at that time. Consequently $\dot{t}$ or $\frac{-Ax}{2a\sqrt{gx}} = 11000 \times \frac{90 + x}{99} \times \frac{-\dot{x}}{a\sqrt{gx}}$ is the fluxion of the time; the correct fluent of which, when $x = 0$, is $11000 \times \frac{180 + \frac{2}{3}d}{99a} \times \sqrt{\frac{d}{g}} = \frac{11000 \times 186 \times 3}{99 \times 4\frac{1}{96}} = 15459''\frac{2}{3}$ nearly, or $4^h 17' 39''\frac{2}{3}$, the whole time of exhausting by a sluice of 1 foot square.

PROBLEM XXVI.

To determine the Time of emptying any Ditch, or Inundation, &c, by a Cut or Notch, from the Top to the Bottom of it.

Let $AB = x$ the variable height of water at any time ;

$AC = b$ the breadth of the cut ;

d = the whole or first depth of water ;

A = the area of the surface of the water in the ditch ;

$g = 16\frac{1}{2}$ feet.



The velocity at any point D, is as $\sqrt{BD}$, that is, as the ordinate DE of a parabola BEC, whose base is AC, and altitude AB. Therefore the velocities at all the points in AB, are as all the ordinates of the parabola. Consequently the quantity of water running through the cut ABGC, in any time, is to the quantity which would run through an equal aperture placed all at the bottom in the same time, as the area of the parabola ABC, to the area of the parallelogram ABGC, that is as 2 to 3.

But $\sqrt{g} : \sqrt{x} :: 2g : 2\sqrt{gx}$ the velocity at AC; therefore $\frac{2}{3} \times 2\sqrt{gx} \times bx = \frac{4}{3}bx\sqrt{gx}$ is the quantity discharged per second through ABGC; and consequently $\frac{4bx\sqrt{gx}}{3A}$ is the velocity per second of the descending sur-

face. Hence then $\frac{4bx\sqrt{gx}}{3A} : -\dot{x} :: 1'' : \frac{-3A\dot{x}}{4bx\sqrt{gx}} = \dot{t}$, the fluxion of the time of descending.

Now when A the surface of the water is constant, or the ditch is equally broad throughout, the correct fluent

of this fluxion gives $t = \frac{3A}{2b\sqrt{g}} \times \frac{\sqrt{d} - \sqrt{x}}{\sqrt{ax}}$ for the general time of sinking the surface to any depth x . And when $x = 0$, this expression is infinite; which shews that the time of a complete exhaustion is infinite.

But if $d = 9$ feet, $b = 2$ feet, $A = 21 \times 1000 = 21000$, and it be required to exhaust the water down to $\frac{1}{16}$ of a foot deep; then $x = \frac{1}{16}$, and the above expression becomes $\frac{3 \times 21000}{4 \times 4\frac{1}{16}} \times \frac{3 - \frac{1}{4}}{\frac{3}{4}} = 14400''$, or just 4 hours for that time. And if it be required to depress it 8 feet, or till 1 foot depth of water remain in the ditch, the time of sinking the water to that point will be $43' 38''$.

Again, if the ditch be the same depth and length as before, but 20 feet broad at bottom, and 22 at top; then the descending surface will be a variable quantity, and, by prob. 25, it will be $= \frac{90 + x}{99} \times 22000$; hence

in this case the fluxion of the time, or $\frac{-3Ax}{4bx\sqrt{gx}}$, becomes $\frac{-500}{3b\sqrt{g}} \times \frac{90 + x}{x\sqrt{x}} \dot{x}$; the correct fluent of which is $t = \frac{1000}{3b\sqrt{g}} \times \left(\frac{90 - x}{\sqrt{x}} - \frac{90 - d}{\sqrt{d}} \right)$ for the time of sinking the water to any depth x .

Now when $x = 0$, this expression for the complete exhaustion becomes infinite.

But if $x = 1$ foot, the time t is $42' 56''\frac{1}{2}$.

And when $x = \frac{1}{16}$ foot, the time is $3^h 50' 28''\frac{1}{2}$.

PROBLEM XXVII.

To determine the Time of filling the Ditches of a Fortification 6 Feet deep with Water, through the Sluice of a Trunk of 3 Feet square, the Bottom of which is level with the Bottom of the Ditch, and the Height of the supplying Water is 9 Feet above the Bottom of the Ditch.

Let ACDB represent the area of the vertical sluice, being a square of 9 square feet, and AB level with the bottom of the ditch. And suppose the ditch filled to any height AE, the surface being then at EF.



Put $a = 9$ the height of the head or supply ;

$$b = 3 = AB = AC ;$$

$$g = 16\frac{1}{2} ;$$

A = the area of a horizontal section of the ditches ;

$x = a - AE$, the height of the head above EF.

Then $\sqrt{g} : \sqrt{x} :: 2g : 2\sqrt{gx}$ the velocity with which the water presses through the part AEFB ; and therefore $2\sqrt{gx} \times \text{AEFB} = 2b\sqrt{gx}(a - x)$ is the quantity per second running through AEFB. Also, by the last problem, the quantity running per second through ECDF is $2\sqrt{gx} \times \frac{2}{3} \text{ECDF} = \frac{4}{3}b\sqrt{gx}(b - a + x)$. Consequently the sum of the two, or $\frac{2}{3}b\sqrt{gx}(a + 2b - x)$ is the quantity per second running in by the whole sluice ACDB.

Hence then $\frac{2}{3}b\sqrt{gx} \times \frac{a + 2b - x}{A} = v$ is the rate or velocity per second with which the water rises in the ditches ; and so $v : -\dot{x} :: 1'' : \dot{t} = -\frac{\dot{x}}{v} = \frac{-3A}{2b\sqrt{g}} \times$

$\frac{x^{-\frac{1}{2}} \dot{x}}{c-x}$ the fluxion of the time of filling to any height AE, putting $c = a + 2b$.

Now when the ditches are of equal width throughout, A is a constant quantity, and in that case the correct fluent of this fluxion is $t = \frac{3A}{2b\sqrt{gc}} \times \log. \left(\frac{\sqrt{c} + \sqrt{a}}{\sqrt{c} - \sqrt{a}} \times \frac{\sqrt{c} - \sqrt{x}}{\sqrt{c} + \sqrt{x}} \right)$ the general expression for the time of filling to any height AE, or $a - x$, not exceeding the height AC of the sluice. And when $x = AC = a - b = d$ suppose, then $t = \frac{3A}{2b\sqrt{gc}} \times \log. \left(\frac{\sqrt{c} + \sqrt{a}}{\sqrt{c} - \sqrt{a}} \cdot \frac{\sqrt{c} - \sqrt{d}}{\sqrt{c} + \sqrt{d}} \right)$ is the time of filling to CD the top of the sluice.

Again, for filling to any height GH above the sluice, x denoting as before $a - AG$ the height of the head above GH, $2\sqrt{gx}$ will be the velocity of the water through the whole sluice AD; and therefore $2b^2\sqrt{gx}$ the quantity per second, and $\frac{2b^2\sqrt{gx}}{A} = v$ the rise per second of the water in the ditches; consequently $v : -\dot{x} :: I'' : \dot{t} = -\frac{\dot{x}}{v} = \frac{-A}{2b^2\sqrt{g}} \times \frac{\dot{x}}{\sqrt{x}}$ the general fluxion of the time; the correct fluent of which, being 0 when $x = a - b = d$, is $t = \frac{A}{b^2\sqrt{g}} (\sqrt{d} - \sqrt{x})$ the time of filling from CD to GH.

Then the sum of the two times, namely, that of filling from AB to CD, and that of filling from CD to GH, is $\frac{A}{b\sqrt{g}} \left[\frac{\sqrt{d} - \sqrt{x}}{b} + \frac{3}{2\sqrt{c}} \log. \left(\frac{\sqrt{c} + \sqrt{a}}{\sqrt{c} - \sqrt{a}} \cdot \frac{\sqrt{c} - \sqrt{d}}{\sqrt{c} + \sqrt{d}} \right) \right]$ for the whole time required. And, using the numbers in the problem, this becomes $\frac{A}{3\sqrt{g}} \left[\frac{\sqrt{6} - \sqrt{3}}{3} + \frac{3}{2\sqrt{15}} \times \log. \right]$

$$\times \log \left(\frac{\sqrt{15} + \sqrt{9}}{\sqrt{15} - \sqrt{9}} \cdot \frac{\sqrt{15} - \sqrt{6}}{\sqrt{15} + \sqrt{6}} \right)] = 0.0383045635 A,$$

the time in terms of A the area of the length and breadth, or horizontal section of the ditches. And if we suppose that area to be 200000 square feet, the time required will be 7661'', or $2^h 7' 41''$.

And if the sides of the ditch slope a little, so as to be a little narrower at the bottom than at top, the process will be nearly the same, substituting for A its variable value, as in prob. 25 and 26. And the time of filling will be very nearly the same as that above determined.

PROBLEM XXVIII.

But if the Water from which the Ditches are to be filled, is the Tide, which at Low Water is below the Bottom of the Trunk, and rises to 9 Feet above the Bottom of it by a regular Rise of One Foot in Half an Hour; It is required to ascertain the Time of Filling it to 6 Feet high as before in the last Problem.

Let $ACDB$ represent the fluice; and when the tide has risen to any height GH , below CD the top of the fluice, without the ditches, let EF be the mean height of the water within. And



Put $b = 3 = AB = AC$;

$$g = 16\frac{1}{12};$$

A = horizontal section of the ditches;

$$x = AG;$$

$$z = AE.$$

P 4

Then

Then $\sqrt{g} : \sqrt{EG} :: 2g : 2\sqrt{g(x-z)}$ the veloc. of the water through AEFB;

and $\sqrt{g} : \sqrt{EG} :: \frac{4}{3}g : \frac{4}{3}\sqrt{g(x-z)}$ the mean veloc. thro' EGHF; theref. $2bz\sqrt{g(x-z)}$ is the quantity per second thro' AEFB; and $\frac{4}{3}b(x-z)\sqrt{g(x-z)}$ is the same through EGHF;

conseq. $\frac{2}{3}b\sqrt{g} \times (2x+z)\sqrt{x-z}$ is the whole thro' AGHB per sec. This quantity divided by the surface A, gives $\frac{2b\sqrt{g}}{3A} \times (2x+z)\sqrt{x-z} = v$ the velocity per second with which EF, or the surface of the water in the ditches, rises.

$$\text{Theref. } v : z :: 1'' : t = \frac{\dot{z}}{v} = \frac{3A}{2b\sqrt{g}} \times \frac{\dot{z}}{(2x+z)\sqrt{x-z}}.$$

But, as GH rises uniformly 1 foot in 30' or 1800'', theref.

$1 : AG :: 1800'' : 1800x = t$ the time the tide is in rising

$$\text{thro' AG; conseq. } \dot{t} = 1800\dot{x} = \frac{3A}{2b\sqrt{g}} \times \frac{\dot{z}}{(2x+z)\sqrt{x-z}},$$

or $m\dot{z} = (2x+z)\sqrt{x-z} \cdot \dot{x}$ is the fluxional equation expressing the relation between x and z ; when $m =$

$$\frac{A}{1200b\sqrt{g}} = \frac{3200}{231} \text{ or } 13\frac{197}{231} \text{ when } A = 200000 \text{ square feet.}$$

Now to find the fluent of this equation, assume

$$z = Ax^{\frac{5}{2}} + Bx^{\frac{8}{2}} + Cx^{\frac{11}{2}} + Dx^{\frac{14}{2}} \&c. \text{ So shall}$$

$$\sqrt{x-z} = x^{\frac{1}{2}} - \frac{A}{2}x^{\frac{3}{2}} - \frac{A^2+4B}{8}x^{\frac{5}{2}} - \frac{A^3+4AB+8C}{16}x^{\frac{7}{2}} \&c,$$

$$2x+z = 2x + Ax^{\frac{5}{2}} + Bx^{\frac{8}{2}} + Cx^{\frac{11}{2}} \&c,$$

$$(2x+z)\sqrt{x-z} = 2x^{\frac{3}{2}} - \frac{3A^2}{4}x^{\frac{5}{2}} - \frac{A^3+6AB}{4}x^{\frac{7}{2}} \&c,$$

$$\text{and } m\dot{z} = \frac{5}{2}mA\dot{x}x^{\frac{3}{2}} + \frac{8}{2}mB\dot{x}x^{\frac{6}{2}} + \frac{11}{2}mC\dot{x}x^{\frac{9}{2}} + \frac{14}{2}mD\dot{x}x^{\frac{12}{2}} \&c,$$

Then

Then equate the coefficients of the like terms,
fo. shall and consequently

$$\frac{5}{2}mA = 2,$$

$$A = \frac{4}{5m},$$

$$\frac{3}{2}mB = 0,$$

$$B = 0,$$

$$\frac{11}{2}mC = -\frac{3}{4}A^2,$$

$$C = -\frac{24}{275m^3},$$

$$\frac{14}{2}mD = -\frac{1}{4}A^3 - \frac{3}{2}AB,$$

$$D = -\frac{16}{875m^4},$$

&c;

&c.

Then these values of A, B, C, &c, substituted in the assumed value of z, give

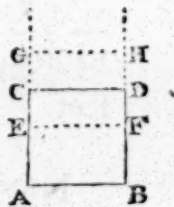
$$z = \frac{4}{5m}x^{\frac{5}{2}} - \frac{24}{275m^3}x^{\frac{11}{2}} - \frac{16}{875m^4}x^{\frac{14}{2}} \&c;$$

$$\text{or } z = \frac{4}{5m}x^{\frac{5}{2}} \text{ very nearly.}$$

And when $x = 3 = AC$, then $z = .886$ of a foot, or $10\frac{2}{3}$ inches, $= AE$, the height of the water in the ditches when the tide is at CD or 3 feet high without, or in the first hour and half of time.

Again, to find the time, after the above, when EF arrives at CD, or when the water in the ditches arrives as high as the top of the sluice.

The notation remaining as before,
then $2bz\sqrt{g(x-z)}$ per sec. runs thro' AF,
and $\frac{2}{3}b(3-z)\sqrt{g(x-z)}$ per sec. thro' ED;
theref. $\frac{2}{3}b\sqrt{g} \times (6+z)\sqrt{x-z}$ is the
whole per sec. through AD.



conseq. $\frac{2b\sqrt{g}}{3A} \times (6+z)\sqrt{x-z} = v$ the veloc. per sec.
of the point E;

and theref. $v : z :: 1'' : t = \frac{z}{v} = \frac{3A}{2b\sqrt{g}} \times \frac{z}{(6+z)\sqrt{x-z}}$
 $= 1800 \dot{x}$, or $m\dot{z} = (6+z)\sqrt{x-z} \cdot \dot{x}$, where m
 $= \frac{1200}{231}$ as before.

Assume

Assume $z = Ax^{\frac{3}{2}} + Bx^{\frac{4}{2}} + Cx^{\frac{5}{2}} + Dx^{\frac{6}{2}} \&c.$ So shall

$$\sqrt{x - z} = x^{\frac{1}{2}} - \frac{A}{2}x^{\frac{3}{2}} - \frac{A^2 + 4B}{8}x^{\frac{5}{2}} - \frac{A^3 + 4AB + 8C}{16}x^{\frac{7}{2}} \&c.$$

$\&c, 6 + z = 6 + Ax^{\frac{3}{2}} + Bx^{\frac{4}{2}} + Cx^{\frac{5}{2}} \&c, (6 + z)$

$\cdot \sqrt{x - z} \cdot \dot{x} = 6x^{\frac{1}{2}}\dot{x} - 3Ax^{\frac{3}{2}}\dot{x} - (\frac{3}{4}A^2 + 3B)x^{\frac{5}{2}}\dot{x} \&c,$

$m\dot{z} = \frac{3}{2}mA x^{\frac{1}{2}}\dot{x} + \frac{4}{2}mB x^{\frac{2}{2}}\dot{x} + \frac{5}{2}mC x^{\frac{3}{2}}\dot{x} \&c.$

Then, equating the like terms, $\&c,$ we have $A = \frac{4}{m},$

$B = -\frac{6}{m^2}, C = \frac{12}{5m^3}, D = \frac{4}{3m^2} \text{ nearly, } E = -\frac{36}{7m^3}$

nearly, $\&c.$

Hence $z = \frac{4}{m}x^{\frac{3}{2}} - \frac{6}{m^2}x^2 + \frac{12}{5m^3}x^{\frac{5}{2}} + \frac{4}{3m^2}x^3 - \frac{36}{7m^3}x^{\frac{7}{2}} \&c.$

Or $z = \frac{4}{m}x^{\frac{3}{2}} \text{ nearly.}$

But, by the first process, when $x = 3, z = .886;$ which substituted for them, we have $z = .886,$ and the series $= 1.36;$ therefore the correct fluents are

$$z - .886 = -1.36 + \frac{4}{m}x^{\frac{3}{2}} - \frac{6}{m^2}x^2 \&c,$$

$$\text{or } z + .474 = \frac{4}{m}x^{\frac{3}{2}} - \frac{6}{m^2}x^2 \&c.$$

And when $z = 3 = AC,$ it gives $x = 5.463$ for the height of the tide without, when the ditches are filled to the top of the sluice, or 3 feet high; which answers to $2^h 43' 53''.$

Lastly, to find the time of rising the remaining 3 feet above the top of the sluice; let

$x = CG$ the height of the tide above $CD,$
 $z = CE$ the height in the ditches above $CD;$
 and the other dimensions as before.



Then

Then $\sqrt{g} : \sqrt{EG} :: 2g : 2\sqrt{g(x-z)}$ = the velocity with which the water runs through the whole sluice AD; consequently $AD \times 2\sqrt{g(x-z)} = 18\sqrt{g(x-z)}$ is the quantity per second running through the sluice, and $\frac{18\sqrt{g}}{A}\sqrt{x-z} = v$ the velocity of z , or the rise of the water in the ditches, per second; hence $v : \dot{z} :: 1'' : \dot{t} = \frac{\dot{z}}{v} = \frac{A}{18\sqrt{g}} \times \frac{\dot{z}}{\sqrt{x-z}} = 1800\dot{x}$, and $n\dot{z} = \dot{x}\sqrt{x-z}$ is the fluxional equation; where $n = \frac{A}{180^2\sqrt{g}} = \frac{m}{9}$

$$= \frac{3200}{2079}.$$

To find the fluent,

$$\text{Assume } z = Ax^{\frac{3}{2}} + Bx^{\frac{4}{2}} + Cx^{\frac{5}{2}} + Dx^{\frac{6}{2}} \&c.$$

$$\text{Then } x - z = x - Ax^{\frac{3}{2}} - Bx^{\frac{4}{2}} - Cx^{\frac{5}{2}} \&c,$$

$$\dot{x}\sqrt{x-z} = x^{\frac{1}{2}}\dot{x} - \frac{A}{2}x^{\frac{3}{2}}\dot{x} - \frac{A^2 + 4B}{8}x^{\frac{5}{2}}\dot{x} \&c,$$

$$n\dot{z} = \frac{3}{2}nAx^{\frac{1}{2}}\dot{x} + \frac{4}{2}nBx^{\frac{3}{2}}\dot{x} + \frac{5}{2}nCx^{\frac{5}{2}}\dot{x} \&c,$$

Then equating the like terms gives

$$A = \frac{2}{3n}, B = \frac{-1}{6n^2}, C = \frac{1}{90n^3}, D = \frac{-1}{810n^4}, \&c.$$

$$\text{Hence } z = \frac{2}{3n}x^{\frac{3}{2}} - \frac{1}{6n^2}x^2 + \frac{1}{90n^3}x^{\frac{5}{2}} - \frac{1}{810n^4}x^3 \&c.$$

But, by the second case, when $z = 0$, $x = 2.463$; which being used in the series, it is 1.273; therefore the correct fluent is $z = -1.273 + \frac{2}{3n}x^{\frac{3}{2}} - \frac{1}{6n^2}x^2 \&c,$

or $z + 1.273 = \frac{2}{3n}x^{\frac{3}{2}}$ nearly. And when $z = 3$, $x =$

6.25 or $6\frac{1}{4}$; the heights above the top of the sluice; answering to 6 and $9\frac{1}{4}$ above the bottom of the ditches. That is, for the water to rise to the height of 6 feet within the ditches, it is necessary for the tide to rise to $9\frac{1}{4}$

without, which answers to $4^h 37\frac{1}{2}$ minutes; and so long it would take to fill the ditches 6 feet deep with water, their horizontal area being 200000 square feet.

Moreover, when $x = 6$, then $z = 2.789$ the height above the top of the sluice; to which add 3, the height of the sluice, and the sum 5.789 , or nearly $5\frac{4}{5}$ feet, is the depth of water in the ditches in 4 hours and a half, or when the tide has risen to the height of 9 feet without the ditches.

OF THE
MOTION OF BODIES IN FLUIDS.

PROBLEM XXIX.

To determine the Force of Fluids in Motion. And the Circumstances attending Bodies moving in Fluids.

1. It is evident that the resistance to a plane, moving perpendicularly through an infinite fluid, at rest, is equal to the pressure or force of the fluid upon the plane at rest, and the fluid moving with the same velocity, and in the contrary direction, to that of the plane in the former case. But the force of the fluid in motion, must be equal to the weight or pressure, which generates that motion; and which, it is known, is equal to the weight or pressure of a column of the fluid, whose base is equal to the plane, and its altitude equal to the height through which a body must fall, by the force of gravity, to acquire the velocity of the fluid: and that altitude is, for the sake of brevity, called the altitude due to the velocity. So that, if a denote the area of the plane, v the velocity, and n the specific gravity of the fluid; then, the altitude due to the velocity v being $\frac{v^2}{4g}$, the whole resistance, or motive force m , will be $a \times n \times \frac{v^2}{4g} = \frac{anv^2}{4g}$; g being $16\frac{1}{2}$ feet. And hence, *cæteris paribus*, the resistance is as the square of the velocity.

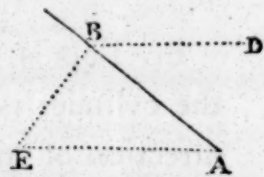
2. This ratio, of the square of the velocity, may be otherwise derived thus. The force of the fluid in motion, must be as the force of one particle multiplied by the

the

the number of them: but the force of a particle is as its velocity; and the number that strikes the plane in a given time, is also as the velocity; therefore the whole force is as $v \times v$ or v^2 , that is, as the square of the velocity.

3. If the direction of motion, instead of being perpendicular to the plane, as above supposed, were inclined to it in any angle, the sine of that angle being s , to the radius 1 ; then the resistance to the plane, or the force of the fluid against the plane, in the direction of the motion, as assigned above, will be diminished in the triplicate ratio of radius to the sine of the angle of inclination, or in the ratio of 1 to s^3 .

For, AB being the direction of the plane, and BD that of the motion, making the angle ABD , whose sine is s ; the number of particles, or quantity of the fluid, striking the plane, will be diminished



in the ratio of 1 to s , or of radius to the sine of the angle B of inclination; and the force of each particle will also be diminished in the same ratio of 1 to s ; so that, on both these accounts, the whole resistance will be diminished in the ratio of 1 to s^2 , or in the duplicate ratio of radius to the sine of the said angle. But again, it is to be considered that this whole resistance is exerted in the direction BE perpendicular to the plane; and any force in the direction BE , is to its effect in the direction AE , parallel to BD , as AE to BE , that is as 1 to s . So that finally, on all these accounts, the resistance in the direction of motion, is diminished in the ratio of 1 to s^3 , or in the triplicate ratio of radius to the sine of inclination. Hence, comparing this with article 1, the whole resistance, or the motive force on the plane, will be $m =$

$$\frac{2n v^2 s^3}{4g}.$$

4. Also

4. Also if w denote the weight of the body, whose plane face a is resisted by the absolute force m ; then the retarding force f , or $\frac{m}{w}$, will be $\frac{anv^2s^3}{4gw}$.

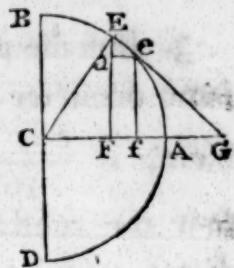
5. And if the body be a cylinder, whose face or end is a , and diameter d , or radius r , moving in the direction of its axis; because then $s = 1$, and $a = pr^2 = \frac{1}{4}pd^2$, where $p = 3.1416$; the resisting force m will be $\frac{npd^2v^2}{16g} = \frac{np r^2 v^2}{4g}$, and the retarding force $f = \frac{npd^2v^2}{16gw} = \frac{np r^2 v^2}{4gw}$.

6. This is the value of the resistance when the end of the cylinder is a plane perpendicular to its axis, or to the direction of motion. But were its face a conical surface, or an elliptic section, or any other figure every where equally inclined to the axis, the sine of inclination being s : then, the number of particles of the fluid striking the face being still the same, but the force of each, opposed to the direction of motion, diminished in the duplicate ratio of radius to the sine of inclination, the resisting force m will be $\frac{npd^2v^2s^2}{16g} = \frac{np r^2 v^2 s^2}{4g}$. But if the body were terminated by an end or face of any other form, as a spherical one, or such like, where every part of it has a different inclination to the axis, then a farther investigation becomes necessary, such as in the following proposition.

PROBLEM XXX.

To determine the Resistance of a Fluid to any Body, moving in it, of a Curved End, as a Sphere, or a Cylinder with a Hemispherical End, &c.

1. Let BEAD be a section through the axis CA of the solid, moving in the direction of that axis. To any point of the curve draw the tangent EG, meeting the axis produced in G: also draw the perpendicular ordinates EF, ef indefinitely near to each other; and draw ae parallel to CG. Putting $CF = x$, $EF = y$, $BE = z$, $s = \text{fine } \angle G \text{ to radius } 1$, and $p = 3.1416$; then $2py$ is the circumference whose radius is EF, or the circumference described by the point E, in revolving about the axis CA; and $2py \times Ee$ or $2pyz$ is the fluxion of the surface, or the surface described by Ee, in the said revolution about CA, and which is the quantity represented by a in art. 3 of the last problem: hence $\frac{nv^2s^3}{4g} \times 2pyz$ or $\frac{pnv^2s^3}{2g} \times yz$ is the resistance on that ring, or the fluxion of the resistance to the body, whatever the figure of it may be. And the fluent of which will be the resistance required.



2. In the case of a spherical form, putting the radius CA or CB = r , we have $y = \sqrt{r^2 - x^2}$, $s = \frac{EF}{EG} = \frac{CF}{CE} = \frac{x}{r}$, and yz or $EF \times Ee = CE \times ae = rx$; therefore the general fluxion $\frac{pnv^2}{2g} \times s^3yz$ becomes $\frac{pnv^2}{2g} \times \frac{x^3}{r^3} \times rx = \frac{pnv^2}{2gr^2} \times x^3 \dot{x}$; the fluent of which, or $\frac{pnv^2}{8gr^2} x^4$, is the

resistance to the spherical surface generated by BE. And when x or CF is $= r$ or CA, it becomes $\frac{pnv^2r^2}{8g}$ for the resistance on the whole hemisphere; which is also equal to $\frac{pnv^2d^2}{32g}$, where $d = 2r$ the diameter.

3. But the perpendicular resistance to the circle of the same diameter d or BD, by art. 5 of the preceding problem, is $\frac{pnv^2d^2}{16g}$; which, being double the former, shews that the resistance to the sphere, is just equal to half the direct resistance to a great circle of it, or to a cylinder of the same diameter.

4. Since $\frac{1}{6}pd^3$ is the magnitude of the globe, if N denote its density or specific gravity, its weight w will be $= \frac{1}{6}pd^3N$, and therefore the retardive force f or $\frac{m}{w} = \frac{pnv^2d^2}{32g} \times \frac{6}{pNd^3} = \frac{3nv^2}{16gNd}$; which is also $= \frac{v^2}{4gs}$ by art. 8 of the general theorems in page 169; hence then $\frac{3n}{4Nd} = \frac{1}{s}$, and $s = \frac{N}{n} \times \frac{4}{3}d$, which is the space that would be described by the globe, while its whole motion is generated or destroyed by a constant force which is equal to the force of resistance, if no other force acted on the globe to continue its motion. And if the density of the fluid were equal to that of the globe, the resisting force is such, as, acting constantly on the globe without any other force, would generate or destroy its motion in describing the space $\frac{4}{3}d$, by that accelerating or retarding force.

5. Hence the greatest velocity that a globe will acquire by descending in a fluid, by means of its relative weight in the fluid, will be found by making the resist-

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ing

ing force equal to that weight. For after the velocity is arrived at such a degree, that the resisting force is equal to the weight that urges it, it will increase no longer, and the globe will afterwards continue to descend with that velocity uniformly. Now, N and n being the separate specific gravities of the globe and fluid, $N - n$ will be the relative gravity of the globe in the fluid, and therefore $w = \frac{1}{6} p d^3 (N - n)$ the weight by which it is urged; also $m = \frac{p n v^2 d^2}{32 g}$ is the resistance; consequently $\frac{p n v^2 d^2}{32 g} = \frac{1}{6} p d^3 (N - n)$ when the velocity becomes uniform; from which is found $v = \sqrt{4 g \cdot \frac{4}{3} d \cdot \frac{N - n}{n}}$ for the said uniform or greatest velocity.

And by comparing this form with that in art. 6 of the general theorems in page 169, it will appear that its greatest velocity is equal to the velocity generated by the accelerating force $\frac{N - n}{n}$ in describing the space $\frac{4}{3} d$, or equal to the velocity generated by gravity in freely describing the space $\frac{N - n}{n} \times \frac{4}{3} d$.—If $N = 2n$, or the specific gravity of the globe be double that of the fluid, then $\frac{N - n}{n} = 1 =$ the natural force of gravity; and then the globe will attain its greatest velocity in describing $\frac{4}{3} d$, or $\frac{4}{3}$ of its diameter.—It is farther evident that if the body be very small, it will very soon acquire its greatest velocity, whatever its density may be.

Ex. If a leaden ball, of 1 inch diameter, descend in water, and in air of the same density as at the earth's surface, the three specific gravities being as $11\frac{1}{3}$, and 1, and

and $\frac{1}{2500}$. Then $v = \sqrt{4 \cdot 16 \frac{1}{12} \cdot \frac{4}{36} \cdot 10 \frac{1}{3}} = \frac{1}{9} \sqrt{31 \cdot 193} = 8.5944$ feet, is the greatest velocity per second the ball can acquire by descending in water. And $v = \sqrt{4 \cdot \frac{193}{12} \cdot \frac{4}{36} \cdot \frac{34}{3} \cdot \frac{2500}{3}}$ nearly $= \frac{50}{9} \sqrt{\frac{34 \cdot 193}{3}} = 259.82$ is the greatest velocity it can acquire in air.

But if the globe were only $\frac{1}{100}$ of an inch diameter, the greatest velocities it could acquire, would be only $\frac{1}{10}$ of these, namely $\frac{86}{100}$ of a foot in water, and 26 feet nearly in air. And if the ball were still farther diminished, the greatest velocity would also be diminished, and that in the subduplicate ratio of the diameter of the ball.

PROBLEM XXXI.

To determine the Relations of Velocity, Space, and Time, of a Ball moving in a Fluid in which it is projected with a given Velocity.

1. Let a = the first velocity of projection, x the space described in any time t , and v the velocity then. Now, by art. 4 of the last problem, the accelerative force $f = \frac{3nv^2}{16gNd}$, where N is the density of the ball, n that of the fluid, and d the diameter. Therefore the general equation $v\dot{v} = 2gf\dot{s}$ becomes $v\dot{v} = \frac{-3nv^2}{8Nd}x$,

and hence $\frac{\dot{v}}{v} = \frac{-3n}{8Nd}x = -b\dot{x}$, putting b for $\frac{3n}{8Nd}$.

And the correct fluent of this is $\log. a - \log. v$ or $\log. \frac{a}{v} = bx$. Or, put $c = 2.718281828$, the number

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whose

whose hyp. log. is 1, then is $\frac{a}{v} = c^{bx}$, and the velocity

$$v = \frac{a}{c^{bx}} = ac^{-bx}.$$

2. The velocity v at any time being the c^{-bx} part of the first velocity, therefore the velocity lost in any time will be the $1 - c^{-bx}$ part, or the $\frac{c^{bx} - 1}{c^{bx}}$ part of the first velocity.

Ex. 1. If a globe be projected, with any velocity, in a medium of the same density with itself, and it describe a space equal to $3d$ or 3 of its diameters. Then $x = 3d$, and $b = \frac{3n}{8Nd} = \frac{3}{8d}$; therefore $bx = \frac{9}{8}$, and the velocity lost is $\frac{c^{bx} - 1}{c^{bx}} = \frac{2.08}{3.08}$, or nearly $\frac{2}{3}$ of the projectile velocity.

Ex. 2. If an iron ball of 2 inches diameter were projected with a velocity of 1200 feet per second, to find the velocity lost after moving through 500 feet of air; we should have $d = \frac{2}{12} = \frac{1}{6}$, $a = 1200$, $x = 500$, $N = 7\frac{1}{3}$, $n = .0012$; and therefore $bx = \frac{3nx}{8Nd} = \frac{3 \cdot 12 \cdot 500 \cdot 3 \cdot 6}{8 \cdot 22 \cdot 10000} = \frac{81}{440}$, and $v = \frac{1200}{c^{\frac{81}{440}}} = 993$ feet per second: having lost 207 feet, or nearly $\frac{1}{6}$ of its first velocity.

Ex. 3. If the earth revolved about the sun in a medium as dense as the atmosphere near the earth's surface; and it were required to find the quantity of motion lost in a year. Then, since the earth's mean density is about $4\frac{1}{2}$,
and

and its distance from the sun 12000 of its diameters, we have $24000 \times 3.1416 = 75398$ diameters $= x$, and $bx = \frac{3 \cdot 75398 \cdot 12 \cdot 2}{8 \cdot 10000 \cdot 9} = 7.5398$; hence $\frac{c^{bx} - 1}{c^{bx}} = \frac{2.575}{2.576}$ parts are lost of the first motion in the space of a year, and only the $\frac{1}{2.576}$ part remains.

Ex. 4. If it be required to determine the distance moved, x , when the globe has lost any part of its motion, as suppose $\frac{1}{2}$, and the density of the globe and fluid equal: The general equation gives $x = \frac{1}{6} \times \log. \frac{a}{v} = \frac{8d}{3} \times \log. \text{ of } 2 = 1.8483925d$. So that the globe loses half its motion before it has described twice its diameter.

3. To find the time t , we have $\dot{t} = \frac{\dot{s}}{v} = \frac{\dot{x}}{v} = \frac{c^{bx}\dot{x}}{a}$. Now to find the fluent of this, put $z = c^{bx}$; then is $bx = \log. z$, and $b\dot{x} = \frac{\dot{z}}{z}$, or $\dot{x} = \frac{\dot{z}}{bz}$; consequently $\dot{t}$ or $\frac{c^{bx}\dot{x}}{a} = \frac{z\dot{x}}{a} = \frac{\dot{z}}{ab}$, and hence $t = \frac{z}{ab} = \frac{c^{bx}}{ab}$. But as t and x vanish together, and when $x = 0$, the quantity $\frac{c^{bx}}{ab} = \frac{1}{ab}$, therefore, by correction, $t = \frac{c^{bx} - 1}{ab} = \frac{1}{bv} - \frac{1}{ba} = \frac{1}{b} \left(\frac{1}{v} - \frac{1}{a} \right)$ the time fought; where $b = \frac{3^n}{8Nd}$, and $v = \frac{a}{c^{bx}}$ the velocity.

Ex. If an iron ball of 2 inches diameter were projected in the air with a velocity of 1200 feet per second; and it were required to determine in what time it would

pass over 500 yards or 1500 feet, and what would be its velocity at the end of that time: We should have, as in ex. 2 above, $b = \frac{3 \cdot 12 \cdot 3 \cdot 6}{8 \cdot 22 \cdot 10000} = \frac{1}{2716}$, and $bx = \frac{1500}{2716} = \frac{375}{679}$; hence $\frac{1}{b} = \frac{2716}{1}$, and $\frac{1}{a} = \frac{1}{1200}$, and $\frac{1}{v} = \frac{c^{bx}}{a} = \frac{1.7372}{1200} = \frac{1}{690}$ nearly. Consequently $v = 690$ is the velocity, and $t = \frac{1}{b} \left(\frac{1}{v} - \frac{1}{a} \right) = 2716 \times \left(\frac{1}{690} - \frac{1}{1200} \right) = 1\frac{3}{8}$ seconds is the time required, or $1''$ and $\frac{2}{3}$ nearly.

PROBLEM XXXII.

To determine the Relations of Space, Time, and Velocity, when a Globe descends, by its own Weight, in an infinite Fluid.

The foregoing notation remaining, viz. d = diameter, N and n the density of the ball and fluid, and v , s , t , the velocity, space, and time, in motion; we have $\frac{1}{6}pd^3$ = the magnitude of the ball, and $\frac{1}{6}pd^3(N - n)$ = its weight in the fluid, also $m = \frac{pnd^2v^2}{32g}$ = its resistance

from the fluid; consequently $\frac{1}{6}pd^3(N - n) - \frac{pnd^2v^2}{32g}$ is the force by which the ball is urged; which being divided by $\frac{1}{6}pNd^3$, the quantity of matter moved, gives $f = 1 - \frac{n}{N} - \frac{3nv^2}{16gNd}$ for the accelerative force.

2. Hence

2. Hence $v\dot{v} = 2gfs$, and $\dot{s} = \frac{v\dot{v}}{2gf} = \frac{Nv\dot{v}}{2g(N-n) - \frac{3n}{8d}v^2}$
 $= \frac{1}{b} \times \frac{v\dot{v}}{a - v^2}$, putting $b = \frac{3n}{8Nd}$, and $\frac{1}{a} =$
 $\frac{3n}{2g \cdot 8d(N-n)}$, or $ab = 2g$; the fluent of which is $s =$
 $\frac{1}{2b} \times \log. \frac{a}{a - v^2}$, an expression for the space s in terms
of the velocity v .

3. But now to determine v in terms of s , put $c =$
 2.718281828 ; then since $\log. \frac{a}{a - v^2} = 2bs$, therefore
 $\frac{a}{a - v^2} = c^{2bs}$, or $\frac{a - v^2}{a} = c^{-2bs}$; and hence $v =$
 $\sqrt{a - ac^{-2bs}}$ the velocity sought.

4. The greatest velocity is to be found, as in art. 5 of
prob. 30, by making f or $1 - \frac{n}{N} - \frac{3nv^2}{16gNd} = 0$, which
gives $v = \sqrt{2g \cdot 8d \cdot \frac{N-n}{3n}} = \sqrt{a}$. The same value is
also obtained by making the fluxion of v^2 or $a - ac^{-2bs} =$
 0 . And the same value of v is obtained by making s in-
finite, for then $c^{-2bs} = 0$. But this velocity $\sqrt{a}$ cannot
be attained in any finite time, and it only denotes the
velocity to which the general value of v or $\sqrt{a - ac^{-2bs}}$
continually approaches. It is evident, however, that it
will approximate towards it the faster, the greater b is,
or the less d is; and that, the diameters being very small,
the bodies descend by nearly uniform velocities, which
are directly in the subduplicate ratio of the diameters. See
also art. 5 prob. 30 for other observations on this head.

5. Since c^{-2bs} is the number whose log is $-2bs$, it will be $c^{-2bs} = c^{-e} = 1 - e + \frac{1}{2}e^2 - \frac{1}{2 \cdot 3}e^3 + \frac{1}{2 \cdot 3 \cdot 4}e^4$ &c, putting $e = 2bs = \frac{3ns}{4Nd}$; hence $1 - c^{-e} = e - \frac{1}{2}e^2 + \frac{1}{6}e^3 - \text{&c}$, and $v = \sqrt{a} \times \sqrt{1 - c^{-e}} = \sqrt{ae} \times (1 - \frac{1}{4}e + \frac{5}{96}e^2 - \frac{1}{128}e^3 \text{ &c})$. And when N is very great in respect of n , then, all the terms after the first being very small, v will be nearly $= \sqrt{ae} = \sqrt{4gs \cdot \frac{N-n}{n}} = \sqrt{4gs}$ nearly, that is the velocity freely generated by gravity, as it ought.

6. To find the time t ; we have $\dot{t} = \frac{\dot{s}}{v} = \sqrt{\frac{1}{a}} \times \frac{\dot{s}}{\sqrt{1 - c^{-2bs}}}$. Then to find the fluent of this fluxion, put $z = \sqrt{1 - c^{-2bs}} = \frac{v}{\sqrt{a}}$, or $z^2 = 1 - c^{-2bs}$; hence $z\dot{z} = b\dot{s}c^{-2bs}$, and $\dot{s} = \frac{z\dot{z}}{bc^{-2bs}} = \frac{1}{b} \cdot \frac{z\dot{z}}{1 - z^2}$; consequently $\dot{t} = \frac{1}{b\sqrt{a}} \cdot \frac{z}{1 - z^2}$, and therefore the fluent is $t = \frac{1}{2b\sqrt{a}} \times \log. \frac{1+z}{1-z} = \frac{1}{2b\sqrt{a}} \times \log. \frac{1 + \sqrt{1 - c^{-2bs}}}{1 - \sqrt{1 - c^{-2bs}}}$ $= \frac{1}{2b\sqrt{a}} \times \log. \frac{\sqrt{a} + v}{\sqrt{a} - v}$, which is the general expression for the time.

7. When N is very great in respect of n ; then, as in art. 5, $v = \sqrt{2abs}$, and $\log. \frac{\sqrt{a} + v}{\sqrt{a} - v} = \log. \frac{\sqrt{a} + \sqrt{2abs}}{\sqrt{a} - \sqrt{2abs}} = \log. \frac{1 + \sqrt{2bs}}{1 - \sqrt{2bs}} = 2\sqrt{2bs}$; and therefore $t = \frac{2\sqrt{2bs}}{2b\sqrt{a}} = \sqrt{\frac{2s}{ab}} = \sqrt{\frac{s}{g}}$, the same as the time of descending freely by gravity, as it ought.

Ex.

Ex. If it were required to determine the time and velocity, by descending in air 1000 feet; the ball being of lead, and 1 inch diameter.

Here $N = 11\frac{1}{3}$, $n = \frac{3}{2500}$, $d = \frac{1}{12}$, and $s = 1000$.

$$\text{Hence } a = \frac{2 \cdot 16\frac{1}{12} \cdot \frac{8}{12} \cdot 11\frac{1}{3}}{3 \cdot \frac{3}{2500}} = \frac{2 \cdot 193 \cdot 8 \cdot 34 \cdot 2500}{3 \cdot 3 \cdot 12 \cdot 12 \cdot 3} = \frac{193 \cdot 34 \cdot 50^2}{9 \cdot 27}, \text{ and } b = \frac{3 \cdot \frac{3}{2500}}{8 \cdot 11\frac{1}{3} \cdot \frac{1}{12}} = \frac{3 \cdot 3 \cdot 3 \cdot 12}{8 \cdot 34 \cdot 2500} = \frac{9 \cdot 9}{68 \cdot 50^2};$$

$$\text{conseq. } v = \sqrt{a} \times \sqrt{1 - c^{-2bs}} = \sqrt{\frac{193 \cdot 34 \cdot 50^2}{9 \cdot 27}} \times$$

$$\sqrt{1 - c^{-\frac{81}{85}}} = 203\frac{2}{3} \text{ the velocity. And } t = \frac{1}{2b\sqrt{a}} \times$$

$$\log. \frac{1 + \sqrt{1 - c^{-2bs}}}{1 - \sqrt{1 - c^{-2bs}}} = \sqrt{\frac{34 \cdot 2500}{27 \cdot 193}} \times \log. \frac{1.78383}{0.21617} =$$

8.5236'', the time.

NOTE. If the globe be so light as to ascend in the fluid; it is only necessary to change the signs of the first two terms in the value of f , or the accelerating force, by which it becomes $f = \frac{n}{N} - 1 - \frac{3nv^2}{16gNd}$; and then proceeding in all respects as before.

SCHOLIUM.

To compare this theory, contained in the last four problems, with experiment, I shall here extract the few following numbers from extensive tables of velocities and resistances, resulting from a course of many hundred very accurate experiments, made by me in the course of the year 1786; of which a particular account will be given elsewhere.

In the first column are contained the mean uniform or greatest velocities acquired in air, by globes, hemispheres, cylinders, and cones, all of the same diameter, and the altitude

tude of the cone nearly equal to the diameter also, when urged by the several weights, expressed in avordupois ounces, and standing on the same line with the velocities, each in their proper columns. So, in the first line, the numbers shew that, when the greatest or uniform velocity was accurately 3 feet per second, the bodies were urged by these weights, according as their different ends went foremost; namely, by $\cdot 028$ oz when the vertex of the cone went foremost; by $\cdot 064$ oz when the base of the cone went foremost; by $\cdot 027$ oz for a whole sphere; by $\cdot 045$ oz for a cylinder; by $\cdot 051$ oz for the flat side of the hemisphere; and by $\cdot 020$ oz for the round or convex side of the hemisphere. Also, at the bottom of all, are placed the mean proportions of the resistances of these figures, in the nearest whole numbers. Note, the common diameter of all the figures, was $6\cdot 375$ or $6\frac{3}{8}$ inches; so that the area of the circle of that diameter is just 32 square inches, or $\frac{2}{9}$ of a square foot; and the altitude of the cone was $6\frac{5}{8}$ inches. Also the diameter of the small hemisphere was $4\frac{3}{4}$ inches, and consequently the area of its base is $17\frac{3}{4}$ square inches, or $\frac{1}{8}$ of a square foot nearly.

The mean height of the barometer at the times of making the experiments, was nearly 30·1 inches, and of the thermometer 62° ; and consequently the weight of a cubic foot of air was equal to $1\frac{1}{5}$ oz nearly in those circumstances.

Velocity per sec.	Cone		Whole globe	Cylin- der	Hemisphere		Small Hemif. flat
	vertex	base			flat	round	
feet	oz	oz	oz	oz	oz	oz	oz
3	·028	·064	·027	·045	·051	·020	·028
4	·048	·109	·047	·090	·096	·039	·048
5	·071	·162	·068	·143	·148	·063	·072
6	·098	·225	·094	·205	·211	·092	·103
7	·129	·298	·125	·278	·284	·123	·141
8	·168	·382	·162	·360	·368	·160	·184
9	·211	·478	·205	·456	·464	·199	·233
10	·260	·587	·255	·565	·573	·242	·287
11	·315	·712	·310	·688	·698	·292	·349
12	·376	·850	·370	·826	·836	·347	·418
13	·440	1·000	·435	·979	·988	·409	·492
14	·512	1·166	·505	1·145	1·154	·478	·573
15	·589	1·346	·581	1·327	1·336	·552	·661
16	·673	1·546	·663	1·526	1·538	·634	·754
17	·762	1·763	·752	1·745	1·757	·722	·853
18	·858	2·002	·848	1·986	1·998	·818	·959
19	·959	2·260	·949	2·246	2·258	·922	1·073
20	1·069	2·540	1·057	2·528	2·542	1·033	1·196
Propor. Numb.	126	291	124	285	288	119	140

From this table of resistances, several practical inferences may be drawn. As,

1. That the resistance is nearly as the surface; the resistance increasing but a very little above that proportion in the greater surfaces. Thus, by comparing together the numbers in the 6th and last columns, for the bases of the two hemispheres, the areas of which are in the proportion of $17\frac{3}{4}$ to 32, or as 5 to 9 very nearly; but the numbers in those two columns, expressing the resistances are nearly as 1 to 2, or as 5 to 10, as far as to the velocity

of 12 feet ; after which the resistances on the greater surface increase gradually more and more above that proportion. And the mean resistances are as 140 to 288, or as 5 to $10\frac{2}{7}$. This circumstance therefore agrees nearly with the theory.

2. The resistance to the same surface, is nearly as the square of the velocity ; but gradually increases more and more above that proportion as the velocity increases. This is manifest from all the columns. And therefore this circumstance also nearly agrees with the theory, in small velocities.

3. When the hinder parts of bodies are of different forms, the resistances are different, though the fore parts be alike ; owing probably to the different pressures of the air on the hinder parts. Thus, the resistance to the fore part of the cylinder is less than that on the flat base of the hemisphere, or of the cone ; because the hinder part of the cylinder is more pressed or pushed, by the following air, than those of the other two figures.

4. The resistance on the base of the hemisphere, is to that on the convex side, nearly as $2\frac{2}{3}$ to 1, instead of 2 to 1, as the theory assigns the proportion. And therefore in this particular, the theory is attended with a considerable error.

5. The resistance on the base of the cone, is to that on the vertex, nearly as $2\frac{3}{10}$ to 1, instead of $5\frac{1}{4}$ to 1, as the theory in pa. 223 art. 6 requires it to be. So that the theory in this instance gives less than half the true experimented resistance.

6. Hence we can find the altitude of a column of air, whose pressure shall be equal to the resistance of a body, moving through it with any velocity.

Let

Let a = the area of the section of the body, similar to any of those in the table, perpendicular to the direction of motion ;

r = the resistance to the velocity in the table ; and

x = the altitude sought, of a column of air, whose base is a , and its pressure r .

Then ax = the content of the column in feet,

and $1\frac{1}{3}ax$ or $\frac{6}{5}ax$ its weight in ounces ;

therefore $\frac{6}{5}ax = r$, and $x = \frac{5}{6} \times \frac{r}{a}$ is the altitude sought in feet, namely $\frac{5}{6}$ of the quotient of the resistance of any body divided by its transverse section ; which is a constant quantity for all similar bodies, however different in magnitude, since the resistance r is as the section a , as we have found in art. 1. When $a = \frac{2}{9}$ of a foot, as in all the figures in our table, except the small hemisphere ; then $x = \frac{5}{6} \times \frac{r}{a}$ becomes $x = \frac{15}{4}r$, where r is the resistance in our table to the similar body. If, for example, we take the convex side of the large hemisphere, whose resistance is .634 oz to a velocity of 16 feet per second ; then $r = .634$, and $x = \frac{15}{4}r = 2.3775$ feet, is the altitude of the column of air whose pressure is equal to the resistance on a spherical surface with a velocity of 16 feet. And to compare the above altitude with that which is due to the given velocity, it will be $32^2 : 16^2 :: 16 : 4$ the altitude due to the velocity 16 ; which is near double the altitude that is equal to the pressure. And as the altitude is proportional to the square of the velocity, therefore, in small velocities, the resistance to any spherical surface, is equal to the pressure of a column of air on its great circle, whose altitude is $\frac{19}{32}$ or .594 of the altitude due to its velocity.

7. Hence we may infer the great resistance suffered by military projectiles. For we find in the table that a globe

globe of $6\frac{3}{8}$ inches diameter, which is equal to the size of an iron ball weighing 36 lb, moving with a velocity of only 16 feet per second, meets with a resistance equal to the pressure of $\frac{2}{3}$ of an ounce weight; and therefore, computing only according to the square of the velocity, the least resistance that such a ball would meet with, when moving with a velocity of 1600 feet, would be equal to the pressure of 417 lb, and that independent of the pressure of the atmosphere itself on the fore part of the ball, which would be 480 lb more, as there would be no pressure from the atmosphere on the hinder part, in the case of so great a velocity as 1600 feet per second. So that the whole resistance would not be less than about 900 lb to such a velocity.

8. Having said in the last article that the pressure of the atmosphere is taken entirely off the hinder part of the ball, moving with a velocity of 1600 feet per second; which must happen when the ball moves faster than the particles of air can follow by rushing into the place quitted and left void by the ball, or when the ball moves faster than the air rushes into a vacuum from the pressure of the incumbent air. Let us therefore inquire what this velocity is. Now the velocity with which any fluid issues, depends upon its altitude above the orifice, and is indeed equal to the velocity acquired by a heavy body in falling freely through that altitude. But, supposing the height of the barometer to be 30 inches or $2\frac{1}{2}$ feet, the height of an uniform atmosphere, all of the same density as at the earth's surface, would be $2\frac{1}{2} \times 14 \times 833\frac{1}{3}$ or 29166 feet; therefore $\sqrt{16} : \sqrt{29166} :: 32 : 8\sqrt{29166} = 1366$ feet, which is the velocity sought. And therefore with a velocity of 1600 feet per second, or any velocity above 1366 feet, the ball must continually leave a vacuum behind it, and so must sustain the whole pressure of the atmosphere

atmosphere on its fore part, as well as the resistance arising from the vis inertia of the particles of air struck by the ball.

9. Upon the whole we find that the resistance of the air, as determined by our experiments, differs very widely, both in respect to the quantity of it on all figures, and in respect to the proportions of it on oblique surfaces, from the same as determined by the preceding theory, which is the same as that of Sir Isaac Newton, and most modern philosophers. Neither should we succeed better if we have recourse to the theory given by professor Gravesande, or others, as similar differences and inconsistencies still occur.

We conclude therefore that all the theories of the resistance of the air hitherto given, are very erroneous. And I have only laid down the preceding one, till further experiments on this important subject shall enable us to deduce from them, another, that shall be more consonant to the true phenomena of nature.



F I N I S.

ERRATA.

Note *b*, at the line; denotes counted from the bottom.

Pa. Line.	Correction.	Pa. Line.	Correction.
2	8 for base <i>r.</i> fide	62	set B in the fig.
3	{ In the fig. of Op. hyp. set A for B at the end of	68	4 hyperbola
4		69	7 <i>b</i> inscribed between the hyperbolas
6	2 <i>b</i> 2CD. 2CE	78	9 <i>b</i> $CG^2 - CA^2$
7	13 for BG read EG	89	2 <i>b</i> asymptote
8	7 <i>b</i> BFK	99	1, 4, 5 for E read F
9	8 <i>b</i> for base read cone	104	in the fig. H and L to change places
13	13 straight	120	draw the line AIE in the fig.
14	4 semi-axes	122	set M and K in the fig.
20	3 <i>b</i> or DG. 2IK :: AD. 2CK + AD. AG : ID ²	123	4 EK.KL
22	7 proportional	134	2 <i>b</i> first col. for 12 read 10
25	the fig. is turned	140	10 for 8 <i>lb</i> read 6 <i>lb</i>
56	draw the line AG in the fig.		

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